Regularization from L^1 by convolution

1 Introduction

The point of this note is to provide background for one of my favorite problems. I offer a prize of 1000 USD for the solution of this problem. I do not think that the problem by itself has any importance. It is just pretty.

It is well known that "convolution spreads regularity". For example, if one want to approximate a continuous function on $\mathbb{R}$ by a $\mathcal{C}^\infty$ function, one takes convolution with a $\mathcal{C}^\infty$ function with (small) compact support. However, some regularization is possible even when one takes convolution with a singular measure, and even when convolution is applied to L^1 functions.

For simplicity we consider only convolution on the group $G = \{-1, 1\}^{\mathbb{N}}$ provided with its Haar measure λ.

Given a positive, finite measure μ on G we consider the operator T_μ on $L^1 = L^1(T, \mathrm{d}x)$ given by

$$T_\mu(f)(x) = \int f(x+y)\mathrm{d}\mu(y) \ .$$

2 Simple facts

It is quite a requirement that T_μ has some regularization properties on L^1.

Proposition 1. Assume that for some Orlicz function φ such that $\lim_{u\to\infty} \varphi(u) = \infty$, for all f in L^1 we have

$$\|T_\mu(f)\|_\varphi \le C\|f\|_1 \ . \tag{1}$$

Then for some constant A we have $\mu \le A\lambda$.

Suppose for contradiction that this is not the case, so that there is compact set K with $\mu(K) > 0$ and $\lambda(K) = 0$. Consider an open set $U \subset G$ and a function $f \in L^1$, such that $x \notin U \Rightarrow f = 0$, with $f \ge 0$ and $\|f\|_1 = 1$. Let us denote by μ_K the restriction of μ to K. Then, using Fubini theorem in the first equality,

$$\mu(K) = \int_G T_{\mu_K}(f)\mathrm{d}\lambda = \int f(x+y)\mathrm{d}\lambda(x)\mathrm{d}\mu_K(y) \ .$$

Now, if $f(x+y) \neq 0$ and $y \in K$ we have $x \in U - K$. Therefore we can restrict the last integral to $x \in U - K$. That is, we have shown that

$$\mu(K) \leq \int_{U-K} T_{\mu_K}(f) \mathrm{d}\lambda(x) \leq \int_{U-K} T_\mu(f) \mathrm{d}\lambda \ , \tag{2}$$

Meanwhile, consider a function $g \geq 0$ with $\|g\|_\varphi \leq C$, so that $\int \varphi(g/C) \mathrm{d}\lambda \leq 1$. For each measurable set V from Jensen's inequality we have

$$\varphi(\int_V g \mathrm{d}\lambda / C\lambda(V) \leq 1 \ .$$

Since we assume that $\lim_{u\to\infty} \varphi(u)\mathrm{d}u = \infty$, this implies that for some constant C' depending only on C we have $\int_V f \mathrm{d}\lambda \leq C'\lambda(V)$.. Using this for $V = U - K$ and $g = T_\mu(f)$ gives

$$\mu(K) \leq C'\lambda(U - K) \ .$$

Since U is arbitary, this gives $\mu(K) \leq C'\lambda(K)$, and since K is arbitrary this proves the result. □

The moral is that (1) is a far too stringent requirement, so we shall consider weaker conditions. We define the function

$$\psi_\mu(u) = \sup \left\{ u\lambda(\{T_\mu(f) \geq u\}) \ ; \ f \geq 0, \|f\|_1 = 1 \right\} .$$

Since

$$\|T_\mu(f)\|_1 = \mu(G)\|f\|_1,$$

from Markov inequality we see that

$$\psi_\mu(u) \leq \mu(G) \ .$$

Here is another simple fact.

Proposition 2 If μ is absolutely continuous with respect to λ then

$$\lim_{u\to\infty} \psi_\mu(u) = 0 \ .$$

This is also almost obvious. When $\mu = \mu_1 + \mu_2$, using that $\lambda(\{f_1 + f_2 \geq u\}) \leq \sum_{j=1,2} \lambda(\{f_j \geq u/2\})$ we obtain that $\psi_\mu(u) \leq 2\psi_{\mu_1}(u/2) + 2\psi_{\mu_2}(u/2)$. Moreover the result is trivial if $\mu \leq C\lambda$ because then $\psi_\mu(u) = 0$ for $u \geq C$. □

The interesting phenomenon is that as we shall see it can happen that μ is singular with respect to λ but that $\lim_{u\to\infty} \psi_\mu(0) = 0$. As the following simple fact shows, this does not happen when μ has a finite support.

Theorem 3 If μ has a finite support then $\liminf_{u\to\infty} \psi_\mu \geq \mu(G)$.

We may assume that μ is a probability. The support of μ is finite, so it generates a finite subgroup H of G. Consider then a subgroup H' of G, so that $H+H'$ is a subgroup of G, which is invariant under translations by elements of the support of μ. Thus if f is the indicator of $H'+H$ it is invariant under translations by elements of the support of μ, and thus $T_\mu(f)=f$. Since the measure of $H+H'$ can be as small as we wish, the result should be obvious. □

Here is another simple fact showing that when μ is singular the function ψ_μ cannot decrease too fast.

Porposition 4 If μ is singular then

$$\int_1^\infty \psi_\mu(u)\mathrm{d}u = \infty\ . \tag{3}$$

For a function $g \geq 0$ and a subset V of G we have, for any number $A \geq 0$,

$$\int_V g\mathrm{d}\lambda = \int_0^\infty \P(\{g \geq u\} \cap V)\mathrm{d}u \leq A\P(V) + \int_A^\infty \P(\{g \geq u\})\mathrm{d}u\ .$$

Consequently, if $f \geq 0$ and $\|f\|_1 = 1$,

$$\int_V T_\mu(f)\mathrm{d}\lambda \leq A\P(V) + \int_A^\infty \psi_\mu(u)\mathrm{d}u\ .$$

Using (2) and taking $V = T - U$, we get

$$\mu(K) \leq A\lambda(U-K) + \int_A^\infty \psi_\mu(u)\mathrm{d}u\ .$$

Since we assume that μ is singular, we can find $a > 0$ such that we can find a with $\mu(K) \geq a$ and $\lambda(U-K)$ as small as we wish. We then see that for each A we have $a \leq \int_A^\infty \psi_\mu(u)\mathrm{d}u$. □

3 The problem

Given $0 \leq a \leq 1$ we consider the "biased coin" probabilty μ_a on G. It is the product measure on G that on each factor gives weight $(1+a)/2$ to the point 1 (and weight $(1-a)/2$ to the point -1) so that

$$\mu_a = \Big(\frac{1+a}{2}\delta_1 + \frac{1-a}{2}\delta_{-1}\Big)^{\otimes\mathbb{N}}\ .$$

Here is a simple fact

Proposition 5 Given $0 < a \leq 1$ there exists $C(a) > 0$ such that for $u \geq 2$ we have $\psi_{\mu_a} \geq C(a)/\sqrt{\log u}$.

We simply look at the density of μ_a when we replace G by $\{-1,1\}^n$. On a sequence with k terms equal to 1 this density g is $(1+a)^k(1-a)^{n-k}$, so that

$$\lambda(g \geq (1+a)^k(1-a)^{n-k}) \geq 2^{-n}\binom{n}{k} .$$

We then choose k as close as possible to $(1+a)n/2$. Letting $c^2 = (1+a)^{1+a}(1-a)^{1-a}$ then $(1+a)^k(1-a)^{n-k}$ is about c^n while, using Stirling's formula, $2^{-n}\binom{n}{k}$ is about $1/(c^n\sqrt{n})$. □

Conjecture 6 Given $a > 0$ there exist $C(a) > 0$ such that for $u \geq 2$ we have

$$\psi_{\mu_a}(u) \leq \frac{C(a)}{\sqrt{\log u}} .$$

I do not know if it is true that $\lim_{u\to\infty} \psi_{\mu_a}(u) = 0$. However the following is proved in my paper "A conjecture on convolution operators, and operators from L^1 to a Banach lattice", Israel J. of Math. 68, 1989, 82-88. In view of (3) this is about as fast a decrease as can be expected.

Theorem 6 The probability measure $\mu = \int_0^1 \mu_{exp(-t)}\mathrm{d}t$ satisfies (for $u \geq 10$)

$$\psi_\mu(u) \leq \frac{C \log\log u}{\log u} .$$

Another noteworthy result is that in 2015 Ronen Eldan and James Lee have solved the "Gaussian limiting case" of Conjecture 6 (with a slightly weaker estimate).