EXERCISES

1. Prove Theorems 96 and 97.
2. Prove Theorem 101.
3. Prove Theorems 103 and 104.
4. Give a simple counterexample to show that in general it is not the case that

$$A \cup (B \times C) = (A \times B) \cup (A \times C).$$

5. Is the Cartesian product operation associative? If so, prove it. If not, give a counterexample.
6. Prove that

$$A \times \bigcap B = \bigcap_{C \in B} (A \times C).$$

§ 2.9 Axiom of Regularity. It is difficult to think of a set which might reasonably be regarded as a member of itself. Certainly the set of all men, for example, is not a man and is therefore not a member of itself. Perhaps it might be argued that in intuitive set theory the set of all abstract objects or the set of all sets should provide an example of a set which is a member of itself, but as we saw in the first chapter, the set of all sets is itself a paradoxical concept.

These remarks suggest we take as an axiom:

$$(1) \qquad A \notin A.$$

However, the assumption of (1) would not prohibit the counterintuitive situation of there being distinct sets A and B such that

$$(2) \qquad A \in B \;\&\; B \in A.$$

(If you do not believe (2) is counterintuitive, try to give a simple example of sets A and B satisfying (2).) Furthermore, if we took (2) as an axiom, longer counterintuitive cycles of membership would not be ruled out — like the existence of distinct sets A, B, and C such that

$$(3) \qquad A \in B \;\&\; B \in C \;\&\; C \in A.$$

We prevent such cycles of any length n by adopting an axiom which is, on the assumption of our other axioms, including the axiom of choice, equivalent to the non-existence of infinite descending sequences of sets (i.e., $A_{i+1} \in A_i$). The form of the axiom which we adopt, the *axiom of regularity*, is due to Zermelo [1930], although an essentially equivalent but more complicated axiom was given earlier in von Neumann [1929, p. 231]:*

$$A \neq 0 \rightarrow (\exists x)[x \in A \;\&\; (\forall y)(y \in x \rightarrow y \notin A)].$$

*The essential idea was formulated even earlier in von Neumann [1925, p. 239] and prior to that in Mirimanoff [1917].

This axiom was called by Zermelo the *Axiom der Fundierung*. Intuitively it says that given any non-empty set A there is a member x of A such that the intersection of A and x is empty. The part '$(\forall y)(y \in x \rightarrow y \notin A)$' which expresses that the intersection of A and x is empty has not been replaced by the simpler appearing formula '$A \cap x = 0$' because of the conditional definition of intersection; for if x is an individual, the definition assigns no intuitive meaning to the intersection of x and any other object. When it is clear that x must be a set, we use the simpler formula in proofs.

We now use the axiom of regularity to prove (1) and the negation of (2) as theorems.

THEOREM 105. $A \notin A$.

PROOF. Suppose that A is a set such that $A \in A$. Since $A \in \{A\}$, we then have

(1) $$A \in \{A\} \cap A.$$

By virtue of the axiom of regularity there is an x in $\{A\}$ such that

$$\{A\} \cap x = 0,$$

but since $\{A\}$ is a unit set,

$$x = A,$$

and thus

$$\{A\} \cap A = 0,$$

which contradicts (1). Q.E.D.

THEOREM 106. $-(A \in B \;\&\; B \in A)$.

PROOF. Suppose that $A \in B \;\&\; B \in A$. Then

(1) $$A \in \{A,B\} \cap B \text{ and } B \in \{A,B\} \cap A.$$

By the axiom of regularity there is an x in $\{A,B\}$ such that

$$\{A,B\} \cap x = 0$$

and by Theorem 43

$$x = A \quad \text{or} \quad x = B.$$

Hence

$$\{A,B\} \cap A = 0 \quad \text{or} \quad \{A,B\} \cap B = 0,$$

which contradicts (1). Q.E.D.

The proof of Theorem 106 proceeded exactly as did that of the previous theorem. Proof of the impossibility of a cycle of three or more sets proceeds similarly.

As an example of the kind of theorem for which the axiom of regularity is essential, we may prove a theorem about Cartesian products which may seem intuitively obvious, but which cannot be proved on the basis of only the axioms introduced earlier.

THEOREM 107. $A \subseteq A \times A \rightarrow A = 0.$

PROOF. Since by hypothesis A is a subset of $A \times A$, from the definition of Cartesian product we know that if $z \in A$ then there are elements x and y such that

(1) $$z = \langle x,y \rangle = \{\{x\}, \{x,y\}\}$$

and

(2) $$x \in A \;\&\; y \in A.$$

Suppose now contrary to the theorem that $A \neq 0$. Let us apply the axiom of regularity to $A \cup \bigcup A$; whence there is a non-empty set C such that

$$C \in A \cup \bigcup A$$

and

(3) $$C \cap (A \cup \bigcup A) = 0.$$

That C must be a non-empty set, and not the empty set or an individual, follows from (1) — the elements of both A and $\bigcup A$ must be non-empty sets. Suppose now that $C \in A$. Then by Theorem 62, $C \subseteq \bigcup A$ and since C is non-empty, we must have

$$C \cap \bigcup A \neq 0,$$

which contradicts (3). Thus C must be in $\bigcup A$, but on the basis of (1), there are elements x and y such that

$$C = \{x\} \vee C = \{x,y\}$$

and on the basis of (2), $x,y \in A$, whence in either case

$$C \cap A \neq 0,$$

which also contradicts (3), and proves that our supposition that $A \neq 0$ is false. Q.E.D.

Even though the axiom of regularity has very natural consequences and imposes, as Zermelo remarked in his 1930 paper, a condition which will be satisfied in all practical applications, it is possible to construct systems of set theory which contradict this axiom. Two examples are Lesniewski's system of ontology (for a good account see Slupecki [1955]) and the system of Quine [1940].

EXERCISES

1. Prove that for all sets A, B, and C it is not the case that

$$A \in B \,\&\, B \in C \,\&\, C \in A.$$

2. Prove that if $A = A \times B$ then $A = 0$.

3. Prove that if $A \times B \neq 0$ then there is a C in $A \times B$ such that $(\bigcup C) \cap (A \times B) = 0$.

4. Prove an analogue of Theorem 46 for the following definition of ordered pairs:

$$\langle x,y \rangle = \{x, \{x,y\}\}.$$

§ 2.10 Summary of Axioms. For convenient subsequent reference we summarize here the six non-redundant axioms introduced in this chapter. The union axiom is omitted because it was shown in §2.6 that it follows from the axiom of extensionality, the pairing axiom, and the sum axiom. These six axioms suffice for all developments in Chapter 3, which is concerned with relations and functions.

Axiom of Extensionality:

$$(\forall x)(x \in A \leftrightarrow x \in B) \rightarrow A = B.$$

Axiom Schema of Separation:

$$(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \,\&\, \varphi(x)).$$

Pairing Axiom:

$$(\exists A)(\forall z)(z \in A \leftrightarrow z = x \vee z = y).$$

Sum Axiom:

$$(\exists C)(\forall x)(x \in C \leftrightarrow (\exists B)(x \in B \,\&\, B \in A)).$$

Power Set Axiom:

$$(\exists B)(\forall C)(C \in B \leftrightarrow C \subseteq A).$$

Axiom of Regularity:

$$A \neq 0 \rightarrow (\exists x)[x \in A \,\&\, (\forall y)(y \in x \rightarrow y \notin A)].$$