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INTEGRAL INEQUALITIES OF GRONWALL TYPE FOR PIECEWISE CONTINUOUS FUNCTIONS

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In this paper we generalize the integral inequality of Gronwall and study the continuous dependence of the solution of the initial value problem for nonlinear impulsive integro-differential equations of Volterra type on the initial conditions.

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AMS subject classifications: 26D10.

1. Introduction

In the present paper analogues of Gronwall's inequality for piecewise continuous functions are introduced. The results obtained for these inequalities are applied to finding sufficient conditions for continuous dependence on the initial conditions of the solutions of the initial value problem for nonlinear impulsive integro-differential equations.

The integral inequalities, established in this paper, can successfully be used in the qualitative theory of the impulsive differential equations.

Let us note that the present paper generalizes some results obtained in [2-4].

2. Basic Notations. Auxiliary Assertions

Let $0 \leq t_0 < t_1 < t_2 < \ldots$ and $\lim_{k \to \infty} t_k = \infty$.

Denote by $PC([t_0, \infty), \mathbb{R}_+)$ the set of all functions $u: [t_0, \infty) \to \mathbb{R}_+$, which are piecewise continuous with discontinuity of the first kind at the points t_k $(k \in \mathbb{N})$, $u(t_k + 0) - u(t_k - 0) < \infty$ and $u(t_k) = u(t_k - 0)$.

Lemma 1: (Theorem 16.4, [1]) *Let for* $t \geq t_0$ *the inequality*

$$u(t) \leq a(t) + \int_{t_0}^{t} g(t,s)u(s)ds + \sum_{t_0 < t_k < t} \beta_k(t)u(t_k), \tag{1}$$

hold, where $\beta_k(t)$ $(k \in \mathbb{N})$ *are nondecreasing functions for* $t \geq t_0$, $a \in PC([t_0,\infty),\mathbb{R}_+)$ *is a nondecreasing function,* $u \in PC([t_0,\infty),\mathbb{R}_+)$, *and* $g(t,s)$ *is a continuous nonnegative function for* t, $s \geq t_0$ *and nondecreasing with respect to* t *for any fixed* $s \geq t_0$.

Then, for $t \geq t_0$ *the following inequality is valid:*

$$u(t) \leq a(t) \prod_{t_0 < t_k < t} (1 + \beta_k(t)) \exp\left(\int_{t_0}^{t} g(t,s)ds \right) \tag{2}$$

3. Main Results

Theorem 1: *Let for* $t \geq t_0$ *the inequality*

$$\begin{aligned} u(t) \leq a(t) &+ \int_{t_0}^{t} b(s)u(s)ds + \int_{t_0}^{t} \left(\int_{t_0}^{s} k(s,\tau)u(\tau)d\tau \right) ds \\ &+ \int_{t_0}^{t} \left(\int_{t_0}^{s} \left(\int_{t_0}^{\tau} h(s,\tau,\sigma)u(\sigma)d\sigma \right) d\tau \right) ds + \sum_{t_0 < t_k < t} \beta_k u(t_k) \end{aligned} \tag{3}$$

hold, where $a, u \in PC([t_0,\infty),\mathbb{R}_+)$, a *is nondecreasing,* $b \in C([t_0,\infty),\mathbb{R}_+)$, $k(t,s)$ *and* $h(t,s,\tau)$ *are continuous and nonnegative functions for* t, $s,\tau \geq t_0$ *and* $\beta_k \geq 0$ $(k \in \mathbb{N})$ *are constants.*

Then, the following inequality is valid:

$$\begin{aligned} u(t) \leq a(t) \prod_{t_0 < t_k < t} (1 + \beta_k) \exp\Bigg(&\int_{t_0}^{t} b(s)ds + \int_{t_0}^{t}\int_{t_0}^{s} k(s,\tau)d\tau ds \\ &+ \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau} h(s,\tau,\sigma)d\sigma d\tau ds \Bigg) \end{aligned} \tag{4}$$

Proof: Denote the right-hand side of inequality (3) by $v(t)$. The function $v \in PC([t_0,\infty),\mathbb{R}_+)$ is nondecreasing, $v(t_0) = a(t_0)$, $u(t) \leq v(t)$ for $t \geq t_0$ and it satisfies the inequality

$$\begin{aligned} v(t) \leq a(t) + \int_{t_0}^{t} &\left[b(s) + \int_{t_0}^{s} k(s,\tau)d\tau + \int_{t_0}^{s}\int_{t_0}^{\tau} h(s,\tau,\sigma)d\sigma d\tau \right] v(s)ds \\ &+ \sum_{t_0 < t_k < t} \beta_k v(t_k). \end{aligned} \tag{5}$$

We apply Lemma 1 to inequality (5) for

$$g(t,s) \equiv b(s) + \int_{t_0}^{s} k(s,\tau)d\tau + \int_{t_0}^{s}\int_{t_0}^{\tau} h(s,\tau,\sigma)d\sigma d\tau, \quad \beta_k(t) \equiv \beta_k,$$

and obtain inequality (4). □

Theorem 2: *Let for $t \geq t_0$ the following inequality hold*

$$u(t) \leq a(t) + \int_{t_0}^{t} b(t,s)u(s)ds + \int_{t_0}^{t} \left(\int_{t_0}^{s} k(t,s,\tau)u(\tau)d\tau \right) ds + \sum_{t_0 < t_k < t} \beta_k(t)u(t_k), \tag{6}$$

where $u, a \in PC([t_0,\infty),\mathbb{R}_+)$, a is nondecreasing, $b(t,s)$ and $k(t,s,\tau)$ are continuous and nonnegative functions for $t,s,\tau \geq t_0$ and are nondecreasing with respect to t, $\beta_k(t)$ $(k \in \mathbb{N})$ are nondecreasing for $t \geq t_0$.

Then, for $t \geq t_0$, the following inequality is valid:

$$u(t) \leq a(t) \prod_{t_0 < t_k < t} (1 + \beta_k(t)) \exp \left(\int_{t_0}^{t} b(t,s)ds + \int_{t_0}^{t} \int_{t_0}^{s} k(t,s,\tau)d\tau ds \right) \tag{7}$$

Proof: Denote the right-hand side of inequality (6) by $v(t)$. The function $v \in PC([t_0,\infty),\mathbb{R}_+)$ is nondecreasing, $u(t) \leq v(t)$ and

$$v(t) \leq a(t) + \int_{t_0}^{t} \left[b(t,s) + \int_{t_0}^{s} k(t,s,\tau)d\tau \right] v(s)ds + \sum_{t_0 < t_k < t} \beta_k(t)v(t_k). \tag{8}$$

We apply Lemma 1 to inequality (8) and obtain inequality (7). □

Theorem 3: *Let for $t \geq t_0$ the following inequality hold*

$$u(t) \leq a(t) + \int_{t_0}^{t} b(s) \left[u(s) + \int_{t_0}^{s} k(\tau)u(\tau)d\tau \right] ds + \sum_{t_0 < t_k < t} \beta_k u(t_k), \tag{9}$$

where u, $a \in PC([t_0,\infty),\mathbb{R}_+)$, a is nondecreasing, b, $k \in C([t_0,\infty),\mathbb{R}_+)$, $\beta_k \geq 0$ $(k \in \mathbb{N})$ are constants.

Then, for $t \geq t_0$, the following inequality is valid:

$$\begin{aligned} u(t) \leq a(t) + &\left\{ \int_{t_0}^{t} b(s) \left[a(s) + \int_{t_0}^{s} k(\tau)a(\tau)d\tau \right] ds + \sum_{t_0 < t_k < t} \beta_k a(t_k) \right\} \\ &\times \prod_{t_0 < t_k < t} (1 + \beta_k) \exp \left\{ \int_{t_0}^{t} b(s) \left[1 + \int_{t_0}^{s} k(\tau)d\tau \right] ds \right\}. \end{aligned} \tag{10}$$

Proof: Consider the function defined by the equality

$$v(t) = \int_{t_0}^{t} b(s) \left[u(s) + \int_{t_0}^{s} k(\tau)u(\tau)d\tau \right] ds + \sum_{t_0 < t_k < t} \beta_k u(t_k). \tag{11}$$

The function $v \in PC([t_0, \infty), \mathbb{R}_+)$ is nondecreasing and satisfies the inequalities

$$u(t) \leq a(t) + v(t) \tag{12}$$

and

$$\begin{aligned} v(t) \leq & \int_{t_0}^{t} b(s) \left[a(s) + \int_{t_0}^{s} k(\tau)a(\tau)d\tau \right] ds + \sum_{t_0 < t_k < t} \beta_k a(t_k) \\ & + \int_{t_0}^{t} b(s) \left[v(s) + \int_{t_0}^{s} k(\tau)v(\tau)d\tau \right] ds + \sum_{t_0 < t_k < t} \beta_k v(t_k). \end{aligned} \tag{13}$$

From inequality (13) and Theorem 1 we obtain the inequality

$$\begin{aligned} v(t) \leq & \left\{ \int_{t_0}^{t} b(s) \left[a(s) + \int_{t_0}^{s} k(\tau)a(\tau)d\tau \right] ds + \sum_{t_0 < t_k < t} \beta_k a(t_k) \right\} \\ & \times \prod_{t_0 < t_k < t} (1 + \beta_k) \exp \left\{ \int_{t_0}^{t} b(s) \left[1 + \int_{t_0}^{s} k(\tau)d\tau \right] ds \right\}. \end{aligned} \tag{14}$$

Thus, (10) follows from inequalities (12) and (14). □

Corollary 1: *Let the conditions of Theorem 3 hold for $a(t) \equiv a = const \geq 0$. Then, for $t \geq t_0$, the following inequality is valid:*

$$\begin{aligned} u(t) \leq a & \left\{ 1 + \int_{t_0}^{t} b(s) \left[1 + \int_{t_0}^{s} k(\tau)d\tau \right] ds + \sum_{t_0 < t_k < t} \beta_k \right\} \\ & \times \prod_{t_0 < t_k < t} (1 + \beta_k) \exp \left\{ \int_{t_0}^{t} b(s) \left[1 + \int_{t_0}^{s} k(\tau)d\tau \right] ds \right\}. \end{aligned}$$

4. Application

With the aid of the established inequalities we shall analyze the continuous dependence of the solutions of the initial value problem for impulsive integro-differential equations on the initial data.

Consider the nonlinear impulsive integro-differential equation

$$\dot{x} = f\left(t, x, \int_{t_0}^{t} k(t, s, x(s))ds \right), \quad \text{for } t \neq t_k, \tag{15}$$

$$\Delta x \mid_{t = t_k} = I_k(x(t_k)), \tag{16}$$

with initial condition

$$x(t_0) = x_0, \tag{17}$$

where $\Delta x \mid_{t = t_k} = x(t_k + 0) - x(t_k - 0)$.

Theorem 4: *Let the following conditions hold:*

1. *The function* $f \in C([t_0, \infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and it satisfies the inequality*

$$\mid f(t, x_1, y_1) - f(t, x_2, y_2) \mid \leq g(t) \mid x_1 - x_2 \mid + h(t) \mid y_1 - y_2 \mid, \quad x_1, x_2, y_1, y_2 \in \mathbb{R},$$

where $g, h \in C([t_0, \infty), \mathbb{R}_+)$.

2. *The function* $k \in C([t_0, \infty) \times [t_0, \infty) \times \mathbb{R}, \mathbb{R})$ *and it satisfies the inequality*

$$\mid k(t, s, x_1) - k(t, s, x_2) \mid \leq m(t, s) \mid x_1 - x_2 \mid, \quad x_1, x_2 \in \mathbb{R}$$

where $m \in C([t_0, \infty) \times [t_0, \infty), \mathbb{R}_+)$.

3. *The functions* $I_k(x): \mathbb{R} \to \mathbb{R}$ $(k \in \mathbb{N})$ *satisfy the inequality*

$$\mid I_k(x_1) - I_k(x_2) \mid \leq \beta_k \mid x_1 - x_2 \mid, \quad x_1, x_2 \in \mathbb{R},$$

where $\beta_k = const > 0$.

4. *For each point* $x_0 \in \mathbb{R}$, *the initial value problem* (15), (16), (17) *has a solution* $x(t; t_0, x_0)$ *for* $t \geq t_0$.

Then, the solutions of equation (15), (16) *depend continuously on the initial conditions, i.e., for any number* $\epsilon > 0$, *there exists a number* $\delta > 0$ *such that for* $\mid x_0 - y_0 \mid < \delta$ *the inequality*

$$\mid x(t; t_0, x_0) - x(t; t_0, y_0) \mid < \epsilon$$

holds for $t \in [t_0, T]$, $T = const > t_0$, $T < \infty$.

Proof: Let $\epsilon > 0$ be an arbitrary number. Consider the function $u(t) = \mid x(t; t_0, x_0) - x(t; t_0, y_0) \mid$, which by the condition of Theorem 4 satisfies the inequality

$$\begin{aligned} u(t) \leq & \mid x_0 - y_0 \mid \\ & + \int_{t_0}^{t} \left[g(s)u(s) + h(s) \int_{t_0}^{s} \mid k(s, \tau, x(\tau; t_0, x_0)) - k(s, \tau, x(\tau; t_0, y_0)) \mid d\tau \right] \\ & + \sum_{t_0 < t_k < t} \beta_k u(t_k) \end{aligned} \tag{18}$$

$$\leq \mid x_0 - y_0 \mid + \int_{t_0}^{t} g(s)u(s)ds + \int_{t_0}^{t} h(s)m(s, \tau)u(\tau)d\tau ds + \sum_{t_0 < t_k < t} \beta_k u(t_k).$$

From inequality (18), by Theorem 2 we obtain the inequality

$$u(t) \leq |x_0 - y_0| \prod_{t_0 < t_k < t} (1 + \beta_k) \exp\left\{ \int_{t_0}^{t} g(s)ds + \int_{t_0}^{t} h(s)m(s,\tau)d\tau ds \right\}. \tag{19}$$

We choose $\delta > 0$ such that

$$0 < \delta < \epsilon \left\{ \prod_{t_0 < t_k < T} (1 + \beta_k) \exp\left[\int_{t_0}^{T} g(s)ds + \int_{t_0}^{T} h(s)m(s,\tau)d\tau ds \right] \right\}^{-1}. \tag{20}$$

Inequalities (19) and (20) yield the assertion of Theorem 4. □

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