66.36 Approximating Square Roots

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which differs from $2\pi/11$ by approximately 1·3%. Perhaps some interested reader could verify that

$$\frac{2\pi}{n} \approx \cos^{-1}\left(\frac{(n-4)\sqrt{3}}{2\sqrt{(n^2-2n+4)}}\right) - \cos^{-1}\left(\frac{n\sqrt{3}}{2\sqrt{(n^2-2n+4)}}\right)$$

and find the relative error in general?

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Number theory textbooks contain proofs of the irrational nature of the square roots of numbers which are not perfect squares. Some continue to mention various methods of obtaining rational approximations to these irrational square roots. However, I have recently discovered a method which may be of interest to *Gazette* readers.

Consider first an example. An approximation to $\sqrt{2}$ is 7/5. So

$$\left(\frac{7}{5} - \sqrt{2}\right)^2 \approx 0$$

$$\therefore \quad \frac{49}{25} - \frac{14}{5}\sqrt{2} + 2 \approx 0$$

$$\therefore \quad \sqrt{2} \approx \frac{99}{70} = 1{\cdot}41426.$$

Alternatively

$$\left(\frac{7}{5} - \sqrt{2}\right)^4 \approx 0$$

$$\therefore \quad \left(\frac{7}{5}\right)^4 - 4\left(\frac{7}{5}\right)^3\sqrt{2} + 6\left(\frac{7}{5}\right)^2 2 - 4\left(\frac{7}{5}\right)2\sqrt{2} + 4 \approx 0$$

$$\therefore \quad \sqrt{2} \approx \frac{\left(\frac{7}{5}\right)^4 + 12\left(\frac{7}{5}\right)^2 + 4}{4\left(\frac{7}{5}\right)^3 + 8\left(\frac{7}{5}\right)} = \frac{19\,601}{13\,860} = 1{\cdot}414213564.$$

In general if n is an integer which is not a perfect square and m is an integer with

$$m < \sqrt{n} < m + 1,$$

then we can take a first rational approximation, r, to $\sqrt{n}$, again choosing r between m and $m + 1$.
Then

$$r - \sqrt{n} = f \quad |f| < 1$$

$$\therefore \quad (r - \sqrt{n})^{2k} = f^{2k} \to 0 \quad \text{as } k \to \infty.$$

$$\therefore \quad \left(\sum_{i=0}^{k} \binom{2k}{2i} r^{2i} n^{k-i}\right) - \sqrt{n}\left(\sum_{i=0}^{k-1} \binom{2k}{2i+1} r^{2i+1} n^{k-i-1}\right) \to 0 \quad \text{as } h \to \infty.$$

$$\therefore \quad \frac{\sum_{i=0}^{k} \binom{2k}{2i} r^{2i} h^{k-i}}{\sum \binom{2k}{2i+1} r^{2i+1} n^{k-i-1}} \to \sqrt{n} \quad \text{as } k \to \infty.$$

(a rational)

Using this method, I have found rational approximations for $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$, as well as $\sqrt{2}$.

n	$\sqrt{n}$ (by calculator)	First approximation	Second approximation ($k = 1$)	Third approximation ($k = 2$)
2	1·414213562	$\frac{7}{5} = 1{\cdot}4$	$\frac{99}{70} = 1{\cdot}414285714$	$\frac{19601}{13860}$ $= 1{\cdot}414213564$
3	1·732050808	$\frac{17}{10} = 1{\cdot}7$	$\frac{589}{340} = 1{\cdot}732352941$	$\frac{693721}{400520}$ $= 1{\cdot}732050834$
5	2·236067977	$\frac{11}{5} = 2{\cdot}2$	$\frac{123}{55} = 2{\cdot}2\dot{3}\dot{6}$	$\frac{30254}{13530}$ $= 2{\cdot}236067997$
7	2·645751311	$\frac{13}{5} = 2{\cdot}6$	$\frac{344}{130} = 2{\cdot}646153846$	$\frac{236636}{89440}$ $= 2{\cdot}645751342$

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