(a) A unital Banach algebra, except the algebra of complex numbers, without nontrivial idempotent.
(b) A unital Banach algebra with a nontrivial idempotent.
(Recall that 0 and 1 are called trivial idempotents.)

****

We show that the Banach algebra $C(X)$ has no nontrivial idempotent iff X is connected:
Let $0 \neq f \neq 1$ be an idempotent. Then $X = f^{-1}(\{0\}) \cup f^{-1}(\{1\})$ implies that X is not connected. Conversely if X is disconnected and $X = G_1 \cup G_2$ with open disjoint sets G_1 and G_2, then $f(x) = \begin{cases} 1 & x \in G_1 \\ 0 & x \in G_2 \end{cases}$ is a trivial idempotent of $C(X)$.

Comment. If A is a (not necessarily commutative) Banach algebra with an element $a \in A$ such that $sp(a)$ is not connected, then A has a nontrivial idempotent. (cf. [B&D, Remarks of Prop. 7.9])

Ref.

[**B&D**] F.F. Bonsall, J. Duncan, complet normed algebras, Springer-Verlag,1973.

A Banach algebra generated by idempotents i.e. elements x such that $x^2 = x$.

****

In the following, we show that the Banach algebra $C(X)$, where X is a compact Hausdorff space, with $Card(X) > 1$,is generated by idempotents iff X is totally disconnected.

Recall that a topological space is said to be totally disconnected if for every distinct $x_1, x_2 \in X$, there exist disjoint open sets G_1 and G_2 such that $x_1 \in G_1, x_2 \in G_2$ and $X = G_1 \cup G_2$.

If X is totally disconnected, $x_1 \neq x_2, x_1 \in G_1, x_2 \in G_2, X = G_1 \cup G_2, G_1 \cap G_2 = \emptyset, G_1$ and G_2 are open, then the continuous function $f(x) = \begin{cases} 1 & x \in G_1 \\ 0 & x \in G_2 \end{cases}$ separates x_1 and x_2. So the closed self-adjoint subalgebra generated by idempotent, by the Stone-Weierstrass theorem, is $C(X)$.

Conversely, suppose that $C(X)$ is generated by its idempotents. Let x_1 and x_2 belong to X. By Urysohn's lemma there exists a function $f \in C(X)$ such that $f(x_1) = 1$ and $f(x_2) = 0$. Every element of the self-adjoint subalgebra generated by idempotents is of the form $h = \sum_{i=1}^{k} \lambda_i g_i (\clubsuit)$ for some idempotents g_i and $\lambda_i \in \mathcal{C}$. Hence there is a sequence (h_n) of elements of the form $(\clubsuit)$ such that $h_n \longrightarrow f$ uniformly on X. So $h_n(x_1) \longrightarrow 1$ and $h_n(x_2) \longrightarrow 0$. Therefore there exists a number N such that $|h_N(x_1)| > \frac{1}{2}$ and $|h_N(x_2)| < \frac{1}{2}$. So that $x_1 \in h_N^{-1}(\{z \in \mathcal{C}; |z| > 1\}) = G_1, x_2 \in h_N^{-1}(\{z \in \mathcal{C}; |z| < 1\}) = G_2, X = G_1 \cup G_2, X = G_1 \cap G_2 = \emptyset$. Thus X is totally disconnected.

A compact Hausdorff space X and subalgebras of $C(X)$ satisfying in only three conditions of four following conditions:

(a) uniformly closed,

(b) separating the points of X,

(c) containing constant functions,

(d) closed under complex conjugation.

(a), (b), (c); i.e. a uniform algebra:

Consider a compact subset X of $\mathcal{C}$ and suppose that A is the uniform closure of rational functions with poles out of X.

(a), (b), (d):

With $X = [a, b]$, let A be the set of all polynomials in one variable, but without constant terms.

(b), (c), (d):

With $X = [a, b]$, put A to be the algebra of all polynomials in one variable.

(a), (c), (d): Let $X = [a, b], x_1$ and x_2 are in X and $A = \{f \in C(X); f(x_1) = f(x_2)\}$.

A Banach algebra A such that $Rad(A)$ is a proper subset of the set $\{x; r(x) = 0\}$ of all quasi-nilpotent elements.

****

1.Suppose that H be a Hilbert space with $dimH \geq 2$. Let $x, y \in H - \{0\}$ and $< x, y >= 0$. The norm of rank one operator $(x\overline{\otimes}y)(z) =< z, y > x$ is $\| x \|\| y \| \neq 0$. So $x\overline{\otimes}y \neq 0$. Also $(x\overline{\otimes}y)^2(z) = (x\overline{\otimes}y)(< z, y > x) =< z, y >< x, y > x = 0$ so $(x\overline{\otimes}y)^2 = 0$. Hence it is quasi-nilpotent. But $B(H)$ is semi-simple. Therefore $x\overline{\otimes}y \notin Rad(B(H)) = \{0\}$.

2.Let $A = M_2(\mathcal{C}) \simeq B(\mathcal{C}^2)$. A is a C^*-algebra so $Rad(A) = \{0\}$. The element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has the spectrum $\{0\}$ and so $r(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = 0$. Hence $Rad(A)$ is not equal to $\{x; r(x) = 0\}$.

An algebrically semisimple non-commutative Banach algebra.

****

We show that $B(X)$, the algebra of bounded linear mappings from normed space X into X is semi-simple:

Suppose that $x_0 \neq 0$ is fixed in X. Then $I_{x_0} = \{T \in B(X); Tx_0 = 0\}$ is a left ideal in $B(X)$.

We shall show that it is maximal. Let J be a left ideal properly containing I_{x_0}. Then $Jx_0 = \{Tx_0; T \in J\}$ is a nonzero linear subspace of X which is invariant under each $S \in B(X)$. If $Jx_0 \neq X$, then there exists a nonzero $y \in Jx_0$ and an element $z \in X$ such that $z \notin Jx_0$. If $S \in B(X)$ such that $Sy = z$, then $z \in Jx_0$ for Jx_0 is invariant under all elements of $B(X)$. Thus $Jx_0 = X$. So that there exists $U \in J$ such that $Ux_0 = x_0$. For each $T \in B(X), TU - UT \in I_{x_0}$. Hence $T \in J + I_{x_0} \subseteq J$. Therefore $B(X) = J$. Thus $Rad(B(X)) \subseteq \cap_{0 \neq x \in X} I_x = \{0\}$. Therefore $B(X)$ is algebrically semisimple.

A semisimple commutative Banach algebra with a closed two-sided ideal I such that $\frac{A}{I}$ isn't semisimple.

****

Suppose that A is the algebra $C^m([0,1])$ of all m times continously differentiable complex-valued functions on $[0,1]$ with the norm $\| f \| = \sum_{k=0}^{m} \frac{1}{k!} \sup_{x\in[0,1]} |f^{(k)}(x)|$. Let $I = \{f \in A; f(0) = f'(0) = 0\}$. Then $\frac{A}{I}$ is not semisimple, since assuming $f_\circ$ to be $f_\circ(x) = x$, then $f_\circ^2 \in I$ and so $(f_\circ + I)^2 = f_\circ^2 + I = 0$, hence $r(x) = \lim_n \| (f_\circ + I)^n \|^{\frac{1}{n}} = 0$. Therefore $f_\circ + I \in Rad(\frac{A}{I})$. But $f_\circ + I \neq 0$. So that $\frac{A}{I}$ is not semisimple.

A non-maximal primary ideal in a unital commutative Banach algebra A.

****

Suppose that A is the algebra $C^m([0,1])$ of the complex valued m times continously differentiable functions on $[0,1]$ with the norm $\| f \| = \sum_{k=0}^{m} \frac{1}{k!} \sup_{x\in[0,1]} |f^{(k)}(x)|$. Let $x_0 \in [0,1]$ and $I = \{f \in A; f(x_0) = f'(x_0) = 0\}$. Then I is a closed two-sided ideal contained in only one maximal ideal; i.e. $\{f \in A : f(x_0) = 0\}$. Note that the maximal ideals of A are of the form $I_x = \{f \in A; f(x) = 0\}, x \in [0,1]$.

A conclusion is that $C^m([0,1])$ is not spectral synthesis, i.e. it has a closed two-sided ideal which is not the intersection of maximal ideals containing this ideal.

Comment. The disk algebra contains a nonmaximal prime ideal, namely $\{0\}$.

An (algebrically) simple Banach algebra.

****

In the case commutative, consider the familiar Banach algebra $\mathcal{C}$.
In the non-commutative case, consider the algebra $M_n(\mathcal{C})$ of all $n \times n$ matrices with entries in $\mathcal{C}$. Identifying $M_n(\mathcal{C})$ with $B(\mathcal{C}^n) = K(\mathcal{C}^n)$ we may regard $M_n(\mathcal{C})$ as a noncommutative C^*-algebra.

Suppose that I_{ij} is the matrix with the ij-entry 1 and 0 elswhere. Then $I_{ij}I_{\alpha\beta} = \delta_{j\alpha}I_{i\beta}$, where δ denotes Kronecker's δ. Let Δ be a nontrivial two-sided ideal in $M_n(\mathcal{C})$. There is a nonzero element $A = \sum_{i,j=1}^{n} a_{ij}I_{ij}$ in Δ, hence $a_{rs} \neq 0$ for some $1 \leq r, s \leq n$. But $I_{rs}AI_{sr} = (\sum_{j=1}^{n} a_{rj}I_{rj})I_{sr} = a_{rs}I_{rr} \in \Delta$. Hence $I_{ij} = I_{is}I_{sr}I_{rj} \in \Delta$ for all $1 \leq i, j \leq n$. Therefore $\Delta = M_n(\mathcal{C})$, a contradiction.

A Banach algebra A, a closed subalgebra B of A and an element $a \in A$ such that $sp(A,a) = sp(B,a)$.

****

Let H be a Hilbert space, $A = B(H), a = T$ be a nonzero element of A and also let B be a maximal commutative subalgebra containing T, then by Theorem 15.4 of [B&D, §15. Theorem 4],

$$sp(A,a) = sp(B,a).$$

Ref.

[**B&D**] F.F. Bonsall, J. Duncan, Complete normed algebras, Springer-Verlag, 1973.

(a) A reflexive Banach algebra.

(b) A non-reflexive Banach algebra.

****

(a)$\mathcal{C}^n$ is reflexive. Note that $(\mathcal{C}^n)^{\#\#} = (\mathcal{C}^n)^{\#} = \mathcal{C}^n$ $(n \geq 1)$.

(b) $c_0^{\#\#} = (l^1)^{\#} = l^\infty$ and the inclusion $c_0 \longrightarrow l^\infty$ is proper. Hence c_0 is not reflexive.

An element of a Banach algebra which has no logarithm.

****

Consider the unilateral shift operator u on a separable Hilbert space H, then u is Fredholm of index $\mathrm{nul}u - \mathrm{def}u = 0 - 1 = -1$. If $\pi : B(H) \longrightarrow \dfrac{B(H)}{K(H)}$ is the quotient map and $\pi(u) = e^w$ for some w in the Calkin algebra $\dfrac{B(H)}{K(H)}$, then there exists an element $w' \in B(H)$ with $\pi(w') = w$, so $\pi(u) = e^w = e^{\pi(w')} = \pi(e^{w'})$. Hence $u - e^{w'} \in K(H)$. But $e^{w'}$ is invertible and so $\mathrm{ind}u = \mathrm{ind}(e^{w'}) = 0$, a contradiction.

An algebra can not be normed so that it becomes a Banach algebra.

****

$A = C^\infty([0,1])$, the algebra of all complex valued infinitely many times continuously differentiable functions on $[0,1]$ is semisimple, for $Rad(A) = \cap_{t\in[0,1]}\{f \in C^\infty([0,1]); f(t) = 0\} = 0$. $f \mapsto f'$ is a derivation on A. The Johnson theorem says that 0 is the only derivation on a semisimple Banach algebra (cf. $[B\&D$, Theorem 18.21]). It follows that $A = C^\infty([0,1])$ is not a Banach algebra under any norm.

For a proof based on the Singer-Wermer theorem see [Sak2, Corollary 2.2.4]).

In addition a direct proof can be found in [Aup, Corollary 4.1.12].

This example is due to Šilov([sil]).

Ref.

[**Aup**] B. Aupetit,A primer on spectral theory, Springer-Verlag, 1991.

[**B&D**] F.F. Bonsall, J.Duncan, complet normed algebras, Springer-Verlag, 1973.

[**Sil**] G.E. Silov, On a property of rings of functions, DoKl. AKad. Nauk. SSSR, 58(1974),985-8.

A commutative radical Banach algebra.

****

1. A Banach space with all products taken to be zero. Then every element is quasi invertible.

2.The Banach space $L^1([0,1])$ with the product $(fg)(x) = \int_0^x f(x-y)g(y)dy$ has $f_\circ(t) = t, 0 \leq t \leq 1$ as a generator since $f_\circ^n(t) = \frac{t^{n-1}}{(n-1)!}$, the set of polynomials in one variable is L^p-dense in $C([0,1])$ and $C([0,1])$ is L^p-dense in $L^1([0,1])$ (cf. [Rud2, Theorem 2.14]).
Moreover $\| f_\circ^n \| = \int_0^1 |f_\circ^n(t)|dt = \frac{1}{n!}$, so $r(f_\circ) = \lim_n \| f_\circ^n \|^{\frac{1}{n}} = \lim_n (n!)^{\frac{-1}{n}} = 0$. Therefore this algebra doesn't have any character. Thus it is radical algebra.

Ref.

[**Rud2**] W. Rudin, Real and complex analysis, McGraw-Hill, 1986.

An element x of a Banach algebra such $r(x) <\| x \|$.

****

Consider $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in the C^*-algebra $M_2(\mathcal{C}) \simeq B(\mathcal{C}^2)$. Then $sp(x) = \{0\}$. So $r(x) = 0$. But $\| x \|= 1$ (since its associated operator $T(z_1, z_2) = (z_2, 0)$ has norm 1).

A commutative Banach algebra A with a unique ideal; i.e. $Rad(A)$.

****

Suppose that $H = L^2(0,1)$ with respect to the Lebesgue measure. Then $(Vf)(x) = \int_0^x f(t)dt$ defines an operator $V \in B(H)$ which is called Volterra operator. Clearly the closure A of $\{p(V); p \text{ is a polynomial in } z\}$ in $B(H)$ is the commutative Banach subalgebra of $B(H)$ generated by V and the identity operator I.

An straightforward computation shows that $(V^n f)(x) = \int_0^x \dfrac{(x-t)^{n-1}}{(n-1)!} f(t)dt$ and so $\| V^n \| \leq \frac{1}{(n-1)!}$ and $r(V) = \lim_n \| V^n \|^{\frac{1}{n}} = 0$. Hence $sp(B(H), V) = \{0\}$. But by [Con, VII.Theorem 5.4], sp(A,V) is equal to the polynomially convex hull of $sp(B(H), V)$, hence $sp(A, V) = \{0\}$. But the maximal ideal space of A is homeomorphic to $sp(A, V) = \{0\}$. So the only character on A is $\phi(\lambda) = \lambda$ and $\phi(x) = 0$ for $x \in A - \mathcal{C}$. Since ϕ is continuous, the unique maximal ideal space is $Rad(A) = Ker(\phi) =$ the closure of $\{p(V); p$ is a polynomial in z and $p(0) = 0\}$.

Ref.

[**Con**]J.B. Conway, A course in functional analysis, New York, Springer-Verlag, 1990.

A Banach algebra A that is a topological direct sum (as a Banach space) of a pair of its Banach subalgebras which are isometrically isomorphic to A.

****

Consider $A = l^\infty$. Define $E = \{(x_n) \in l^\infty \ ; \ x_{2n} = 0\}$ and $F = \{(x_n) \in l^\infty \ ; \ x_{2n-1} = 0\}$. Obviously E and F are closed subalgebras of l^∞. Moreover $A = E + F$ and $E \cap F = \{0\}$. So E is a complemented subspace of A with F as a complementary subspace. In addition, $\varphi((x_1, x_2, x_3, \ldots)) = (0, x_1, 0, x_2, 0, x_3, \ldots)$ is an isometrically isomorphism between l^∞ and E. One can similarly define an isomorphism between l^∞ and F. Note that if E is a complemented infinite dimensional subspace of l^∞ then E is isomorphic to l^∞. (cf. [J. Lindenstranss, On complemented subspaces of m. Israel J. Math., 5, 1967, 153-156])

A Banach algebra with a proper dense two-sided ideal.

****

1. $C_c(\mathcal{R}) = \{f \in C_0(\mathcal{R});\ \text{supp}(f) = \text{ the closure of } \{x \in \mathcal{R}; f(x) \neq 0\} \text{ is compact }\}$ is a dense ideal of $C_0(\mathcal{R})$. Note that the function f defined by

$$f(x) = \begin{cases} \frac{1}{1+x} & x \geq 0 \\ \frac{1}{1-x} & x < 0 \end{cases}$$

belongs to $C_0(\mathcal{R}) - C_c(\mathcal{R})$.

2. $A = \{f \in C([0,1]); f(0) = 0\}$ is a closed subalgebra of $C([0,1])$ not containing the constant function 1. So A is a non-unital Banach algebra. Let $f_\circ(t) = t$, $t \in [0,1]$. $I = \{f_\circ g; g \in C[0,1]\}$ is a proper ideal of A (since if

$$h(t) = \begin{cases} t sin\frac{1}{t} & t \in (0,1] \\ 0 & t = 0 \end{cases}$$

and for some $g \in C[0,1], tg(t) = h(t)$ whenever $t \in [0,1]$ then $\lim_{t\to 0} \sin\frac{1}{t} = g(0)$, a contradiction). By the Stone-Weierstrass theorem, each $f \in A$ is the uniform limit of a sequence (p_n) of polynomials with $p_n(0) = 0$. Moreover $t \longmapsto \frac{p_n(t)}{t}$ belongs to $C[0,1]$ and $t\frac{p_n(t)}{t} \longrightarrow f(t)$ uniformly on $[0,1]$. So f belongs to the closure of I. Hence I is dense in A.

A Banach algebra A in which every singular element is a left or right topological divisor of zero.

****

Let $A = B(X)$, the Banach algebra of bounded linear mappings from a Banach space X into X, and $T \in Sing(A)$. For $y \in X$ and $g \in X$, the rank one operator $y\overline{\otimes}g \in B(X)$ is defined by $(y\overline{\otimes}g)(z) = g(z)y$ $(z \in X)$.

If T isn't 1-1, then there exists an element $x \neq 0$ such that $Tx = 0$. So if $f \in X^{\#}$ and $f(x) = 1$, then $T(x\overline{\otimes}f) = Tx\overline{\otimes}f = 0$. So T is a left divisor of zero.

If $T(X) \neq X$ and $T(X)^{-} \neq X$, then by the Hahn-Banach theorem there exists a non-zero functional f such that $f(T(X)) = 0$. Therefore $(x\overline{\otimes}f)T = 0$, for all $x \in X$. Thus T is a right divisor of zero.

Finally if $T(X) \neq X$ and $T(X)^{-} = X$, then there exists a sequence (y_n) in X satisfying $||y_n|| = 1$ and $Ty_n \longrightarrow 0$. If $f \in X^{\#}$ with $||f|| = 1$ and $U_n = y_n\overline{\otimes}f$, then $||U_n|| = ||y_n||\ ||f|| = 1$ and $||TU_n|| = ||Ty_n\overline{\otimes}f|| \leq ||Ty_n||$ and hence $TU_n \longrightarrow 0$. Thus T is a right topological divisor of zero.

Two element a, b of a Banach algebra such that neither $r(ab) \leq r(a)r(b)$ nor $r(a+b) \leq r(a)r(b)$.

****

Consider $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ in $M_2(\mathcal{C}) \simeq B(\mathcal{C}^2)$. Then $sp(a) = \{0\}, sp(b) = \{0\}, sp(a+b) = \{-1, 1\}, sp(ab) = \{0, 1\}$ and we have required inequalities.

A normed algebra with non-open group of invertibles (and so the algebra is not Banach).

****

Let $A = \mathcal{C}[z]^1$, then $Inv(\mathcal{C}[z]) = \mathcal{C} - \{0\}$, hence the elements $p_n(z) = 1 + \frac{z}{n}(n \in \mathcal{N})$ aren't invertible. But $\lim_n p_n(z) = 1 \in Inv(\mathcal{C}[z])$. Therefore $A - Inv(A)$ isn't closed.

[1] The set $\mathcal{C}[z]$ of all polynomials in an indeterminate z with complex coefficients under usual operations on polynomials and with the norm $||p|| = \sup_{|\lambda| \leq 1} |p(\lambda)|$ is a normed algebra.

A commutative Banach algebra whose unit ball isn't norm compact.

****

The unit ball of $C([0,1])$ is not compact with respect to the supremum norm, since if $p_n(x) = x^n$, then $\| p_n \| = 1$ and (p_n) has no convergent subsequence.
It's well-known that a normed space Y is finite dimensional iff $\{y \in Y; \| y \| \leq 1\}$ is compact (cf. [Ker, Theorem 2.5-5]). $C([0,1])$ is infinite dimensional, hence its unit ball is not compact.

Ref.

[Ker] E. Kreyszig, Introductory functional analysis with applications,John Wiley & Sons, 1978.

A normed algebra A whose radical is isomorphic to $\mathcal{C}$.

****

Suppose that $A = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}; a, b, c \in \mathcal{C}\}$. Then A is a subalgebra of $M_2(\mathcal{C}) \simeq B(\mathcal{C}^2)$ and the only its characters are $f(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = a$ and $g(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix})) = c$, since

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

is a basis for A. Therefore $Rad(A) = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}; b \in \mathcal{C}\}$ is isometrically isomorphic to $\mathcal{C}$.

(a) A separable Banach algebra.

(b) A non-separable Banach algebra.

****

(a) $\{r_1 + ir_2; r_1, r_2 \in \mathcal{Q}\}$ is a countable dense subset of $\mathcal{C}$. Hence $\mathcal{C}$ is a separable Banach algebra.

(b) l^∞ isn't separable. In fact if $S = \{a_1, a_2, \cdots,\}$ is a countable set in l^∞, $a_n = (a_n^k)_{k\in\mathcal{N}}$, and $b_n = \begin{cases} 0 & |a_n^n| \geq 1 \\ 2 & |a_n^n| < 1 \end{cases}$, then $b = (b_n) \in l^\infty$ and for all n, $\| b - a_n \|_\infty \geq |b_n - a_n^n| \geq 1$. So that the neighborhood of b with the radius 1 doesn't intersect S. Thus S isn't dense in l^∞.

For another proof see [A&B, Problem 25.7].

Ref.

[**A&B**] C.D. Aliprantis and O. Burkinshaw, Problems in real analysis, Acad. Press, 1990.

Two non-isomorphic Banach algebras with homeomorphically isomorphic invertible groups.

****

1. Let $A_1 = C([-1, \frac{-1}{2}] \cup [\frac{1}{2}, 1])$ and $A_2 = C([0,1] \cup \{2\})$. Since $[0,1] \cup \{2\}$ isn't homeomorphic to $[-1, \frac{-1}{2}] \cup [\frac{1}{2}, 1]$, A_1 isn't isomorphic to A_2. Also the function which sends $x \in Inv(A_1)$ to $y \in G_2$ defined by

$$y(t) = \begin{cases} x(t-1) & t \in [0, \frac{1}{2}] \\ \left[x(\frac{-1}{2})/x(\frac{1}{2})\right] x(t) & t \in [\frac{1}{2}, 1] \\ x(\frac{1}{2})/x(\frac{-1}{2}) & t = 2 \end{cases}$$

is the desired isomorphism.

Ref.

[**Zel**] W. Zelazko, Banach algebras, Elsevier Publishing Company, 1973.

A commutative Banach algebra whose unit ball has no extreme point (and so it isn't the dual space of any Banach space by the Krein-Milman theorem (cf. [Con, Theorem 7.4])).

****

The unit ball of c_0 has no extreme point. For see this, let (x_n) belongs to the ball of c_0. $\lim_n x_n = 0$, so there exists a number N such that for all $n > N$, $|x_n| < \frac{1}{2}$. Let $y_n = z_n = x_n$ for $n \leq N$, and let $y_n = x_n + 2^{-n}$ and $z_n = x_n - 2^{-n}$ for $n > N$, then (y_n) and (z_n) belong to the unit ball of c_0 and $(x_n) = \frac{1}{2}(y_n) + \frac{1}{2}(z_n)$. So (x_n) isn't is not an extreme point.

Ref.

[**Con**]J.B. Conway, A course in functional analysis, New York, Springer-Verlag, 1990.

(i) A singly generated Banach algebra

(ii) A Banach algebra can not be singly generated

****

(i) $C_\circ((0,1])$ is singly generated by the inclusion function $t \longmapsto t$, by the Stone-Weierstrass theorem.

(ii) $C(\Gamma)$, where Γ is the unit circle in plane.

A Banach algebra without any topological divisor of zero.

****

Clearly $\mathcal{C}$ has no topological divisor of zero. In fact $\mathcal{C}$ is the only Banach algebra with this property. (cf. [W. Zelazco, On generalized topological divisors of zero in real m-convex algebras, (1967) 241-244.]).

A commutative Banach algebra A without any minimal ideals.

****

Let $A = \mathcal{A}(\Delta)$[1] , J be a minimal ideal and, for $n \geq 0$, $I_n = \{f \in A \; ; \; f(0) = f'(0) = \ldots = f^{(n)}(0) = 0\}$ (recall $f^{(0)} = f$). Then $(I_n)_{n \geq 0}$ is a strictly decreasing sequense of (primary) ideals. Assuming $0 \neq f \in J$, then $0 \neq z^{n+1} f \in I_n \cap J$. So $I_n \cap J = J$. Hence $(\bigcap_{n=1}^{\infty} I_n) \cap J = J$ and so $J = 0$, since $\bigcap_{n=1}^{\infty} I_n = \{0\}$. Thus $\mathcal{A}$ has no minimal ideal.

[1] Let Δ denote the closed unit disc $\{z \in \mathcal{C}, |z| \leq 1\}$. Suppose that $A(\Delta)$ denoted the set of all elements of $C(\Delta)$ which are analytic on the interior of Δ. $A(\Delta)$ is a closed subalgebra of $C(\Delta)$

Two elements x, y $(xy \neq yx)$ of a Banach algebra A such that $e^x.e^y \neq e^{x+y}$.

****

Consider $A = B(l^2)$ and the unilateral shift operator T on l^2, defined by $T(x_1, x_2, ...) = (0, x_1, x_2, ...)$ and its adjoint $T^*(x_1, x_2, ...) = (x_2, x_3, ...)$. Assuming $\xi_k = (\delta_{kn})_{n \in \mathcal{N}}, k \in \mathcal{N}; < e^T e^{T^*} \xi_1, \xi_1 > = < e^T \xi_1, \xi_1 > = < \xi_1, \xi_1 > = 1$, since $T^* \xi_1 = 0$ and $T\xi_1 = \xi_2$. Also $(T + T^*)(\xi_1) = \xi_2, (T + T^*)^2(\xi_1) = \xi_1 + \xi_3, \ldots$ and so $< e^{T+T^*} \xi_1, \xi_1 > = < \xi_1, \xi_1 > + < \xi_2, \xi_1 > + < \frac{1}{2!}(\xi_1 + \xi_3), \xi_1 > + \ldots > 1$. Hence $e^T.e^{T^*} \neq e^{T+T^*}$.

A reflexive Banach algebra whose dual is also a Banach algebra.

****

The Banach algebra l^p[1], $1 < p < \infty$ has the conjugate l^q, $q = \dfrac{p}{p-1}$, in addition $(l^q)^{\#} = l^p$.

[1] Let (Ω, μ) be a measure space and $L^p(\Omega, \mu)$ for $1 \leq p < \infty$ be the set of all complex valued measurable functions f on Ω (we assume f is equal to g if $f = g$ a.e.$[\mu]$) for which $\|f\|_p = (\int_\Omega |f|^p d\mu)^{\frac{1}{p}} < \infty$. $L^p(\Omega, \mu)$ with the norm $\|.\|_p$ is a Banach space and is a Hilbert space iff $p = 2$. $L^p(\Omega, \mu)$ denoted by $l^p(\Omega)$ if μ is counting measure. In particular, $l^p(\mathcal{N})$ denoted by l^p.

If $1 \leq p < \infty$, then l^p can be regarded as a commutative Banach algebra with coordinatewise multiplication. (For $p > 1$, $\|fg\|_p \leq \|f\|_p \|g\|_p$ is a conclusion of Hőlder inequality.) The l^p, $1 \leq p < \infty$, with the involution $f \longmapsto \overline{f}$ is an involutive Banach algebra.

A Banach algebra A that cannot be a (vector space) direct sum of its radical $Rad(A)$ and a Banach algebra B that is homeomorphically isomorphic with $A/Rad(A)$.

****

Consider the Banach algebra l^2 and the dense subalgebra l_0^2 of l^2 consisting of the sequences which vanish out of a finite set. Let A_0 be the vector space direct sum $l_0^2 \oplus \mathcal{C}$. A_0 is an algebra with $(x,\alpha)(y,\beta) = (xy, 0), x, y \in l^2, \alpha, \beta \in \mathcal{C}$. Also $||(x,\alpha)|| = \max(||x||, |\alpha - \sum_{n=1}^{\infty} x(n)|)$ is a norm on A_0. Let A is the completion of A_0. $Rad(A) = \mathcal{C}(0,1)$. If $(x,\alpha) \in A_0$ and $[x,\alpha]$ denotes the image of (x,α) in $A/Rad(A)$, then $[x,\alpha] \mapsto x$ defines an isometric isomorphism of $A_0/RadA$ into l_0^2 which can be extended to an isometric isomorphism of $A/RadA$ onto l^2. Suppose that there exists a homeomorphic isomorphism of l^2 with a subalgebra A_1 of A. Let ξ_k denotes $\xi_k(n) = \delta_{kn}$ $(k, n \in \mathcal{N})$ and e_k denotes the corresponding element of A_1. Choose a sequence $((x_n, \alpha_n))_{n\in\mathcal{N}}$ in A_0 such that $\lim_n (x_n, \alpha_n) = e_k$ in A. Since $e_k^2 = e_k$, we have $\lim_n (x_n^2, 0) = e_k$. Thus $\lim_n x_n(k) = 0$ or 1 for all $k \in \mathcal{N}$ and also $e_k \in l_0^2$. The elements e_k are pairwise orthogonal idempotents. If $d_n = \sum_{k=1}^{n} \frac{e_k}{k}$ and $t_n = \sum_{k=1}^{n} \frac{\xi_k}{k}$, then (t_n) converges in l^2. But d_n doesn't converge in A, a contradiction.

Ref.

[**Ric**] C.E. Rickart, General theory of Banach algebras, Princeton, Van Nastrand, 1960.

A commutative Banach algebra where 0 is the only nilpotent.

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A C^*-algebra is commutative if and only if it has 0 as its unique nilpotent element. This is due to I. Kaplansky.(cf. [I. Kaplansky, Ring isomorphisms of Banach algebras, Canada. J.Math. 6 (1954), 374-381.])

A non-commutative Banach algebra in which 0 is the only quasi-nilpotent.

****

Let A be the free algebra on two symbols w, v, i.e. the algebra of all finite linear combinations of words in u and v. The set of all such words is countable, $\{w_n\}$, and we take the standard enumeration given by $u, v, u^2, uv, v^2, u^3, u^2v, \ldots$. Let B be the algebra of all infinite series $x = \sum_{n=1}^{\infty} \alpha_n w_n$, where $||x|| = \sum_{n=1}^{\infty} ||\alpha_n| < \infty$. Then B is a non-commutative Banach algebra. Let $x \in B, x \neq 0$, and let α_p be the first non-zero coefficient in the series $\sum_{n=1}^{\infty} \alpha_n w_n$. Then the coefficient of w_p^m in x^m is precisely α_p^m and so $||x^m|| \geq |\alpha_p|^m$ $(m = 1, 2, 3, \ldots)$, $r(x) \geq |\alpha_p| > 0$. Note that B is an infinite dimensional non-commutative Banach algebra in which the set of quasi-nilpotents coincides with the set of nilpotents.

Ref.

J. Duncan and A.W. Tullo, Finite dimensionality, nilpotents and quasi-nilpotents in Banach algebras, Proc. of the Edin. math. Soc., vol 19(Series II), Part 1, 1974.

A non-commutative radical Banach algebra which is an integral domain.

****

Let A be the free algebra on two symbols w, v, i.e. the algebra of all finite linear combinations of words in u and v. The set of all such words is countable, $\{w_n\}$, and we take the standard enumeration given by $u, v, u^2, uv, v^2, u^3, u^2v, \ldots$. Let $\gamma(w_n)$ denote the length of the word w_n, and let C be the algebra of all infinite series $x = \sum_{n=1}^{\infty} \alpha_n w_n$ where $\|x_n\| = \sum \dfrac{|\alpha_n|}{\gamma(w_n)!} < \infty$. Then C is clearly a non-commutative Banach algebra and an integral domain. Let $x \in C$ and let k be a positive integer. We have

$$\begin{aligned}
\|x^k\| &\leq \sum_{n_i} \frac{|\alpha_{n_1}||\alpha_{n_2}| \ldots |\alpha_{n_k}|}{\gamma(w_{n_1} w_{n_2} \ldots w_{n_k})!} \\
&= \sum_{n_i} \frac{\gamma(w_{n_1})! \ldots \gamma(w_{n_k})! |\alpha_{n_1}|}{\{\gamma(w_{n_1}) + \ldots + \gamma(w_{n_k})\}! \gamma(w_{n_1})!} \cdots \frac{|\alpha_{n_k}|}{\gamma(w_{n_k})!} \\
&\leq \frac{1}{k!} \|x\|^k.
\end{aligned}$$

Hence $r(x) = 0$.

Ref.

J. Duncan and A.W. Tullo, Finite dimensionality, nilpotents and quasi-nilpotents in Banach algebras, Proc. of the Edin. math. Soc., vol 19(Series II), Part 1, 1974.

A non-reflexive Banach space isometric with its second conjugate space.

****

For $x = (x_1, x_2, x_3, \ldots)$, let $\|x\| = \sup[\sum(x_{p_i} - x_{p_{i+1}})^2 + (x_{p_{n+1}} - x_{p_1})^2]^{\frac{1}{2}}$ where supremum is over all positive integers n and all finite increasing sequences of at least two positive integers $p_1, p_2, \ldots, p_{n+1}$. Let B be the Banach space of all x for which $\|x\|$ is finite and $\lim_n x_n = 0$. Then B is isometric with $B^{\#\#}$, but is isometric under natural mapping with a closed maximal linear subspace of $B^{\#\#}$. This example is due to R.C. James (cf. [Jam]).

Ref.

[Jam] R.C.James, A non-reflexive Banach space isometric with its second conjugate space, Proc.of.nat.Acad. of sci., Vol 37, No 3, pp. 174-177, 1951.

A Banach algebra A with a Banach subalgebra B and an element $b \in B$ such that $sp(A,b)$ is a proper subset of $sp(B,b)$.

****

Consider $\mathcal{A}(\Delta)$[1] and the isometric isomorphism $f \longmapsto f|_T$, from $\mathcal{A}(\Delta)$ onto the closed subalgebra B of $A = C(T)$ generated by 1 and inclusion $z : T \to \mathcal{C}$ (T is the unit circle). Then $sp(B,z) = sp(\mathcal{A}(\Delta), z) = \Delta$ and $sp(A,z) = T$.

[1]Let Δ denote the closed unit disc $\{z \in \mathcal{C}, |z| \leq 1\}$. Suppose that $A(\Delta)$ denoted the set of all elements of $C(\Delta)$ which are analytic on the interior of Δ. $A(\Delta)$ is a closed subalgebra of $C(\Delta)$.

A Banach algebra with an unbounded approximate identity.

****

Consider l^p as a Banach algebra with coordinatewise operations. Let $e_n = (\underbrace{1,1,1,\ldots,1}_{n},0,0,\ldots)$. Then $\sup_n ||e_n|| = \sup\{\sqrt[p]{n}\ ;\ n \in N\} = \infty$, and for every $x = (\alpha_n) \in l^2$, $\lim_n ||xe_n - x|| = \lim_n (\sum_{k=n+1}^{\infty} |\alpha_k|^p)^{\frac{1}{p}} = 0$. Thus (e_n) is required approximate identity.

A topologically nilpotent Banach algebra. (A Banach algebra A is called topologically nilpotent if the quantity $N_A(n) = \sup\{\|x_1 ... x_n\|^{\frac{1}{n}}$; $x_i \in A$; $\|x_i\| \leq 1$, $1 \leq i \leq n\}$ tends to zero as $n \to \infty$).

****

The Banach algebra $C[0,1]$ with the supremum norm $\|.\|$ and convolution multiplication is topologically nilpotent:

Defining $u \in C([0,1])$ by $u(t) = 1$ $(0 \leq t \leq 1)$, we have $u^n(t) = \frac{t^{n-1}}{(n-1)!}$ $(n = 1, 2, \ldots)$ and so $\|u^n\| = \frac{1}{(n-1)!}$. For arbitrary $f_1, \ldots, f_n \in C([0,1])$, $|f_1 * f_2 * \ldots * f_n)(t)| \leq \|f_1\| \ldots \|f_n\| u^n(t)$. Hence $(\frac{\|f_1 * \ldots * f_n\|}{\|f_1\| \ldots \|f_n\|})^{\frac{1}{n}} \leq \frac{1}{((n-1)!)^{\frac{1}{n}}}$. Now note that $\lim_n \frac{1}{((n-1)!)^{\frac{1}{n}}} = 0$.

Ref.

P.G. Dixon, G. A. Willis, Approximate identities in extensions of topological nilpotent Banach algebras., Proc. Royal of Edin., 122A, 45-52, 1992.

A non-topologically nilpotent Banach algebra.

****

1. The algebra $\mathcal{C}$ of complex numbers.

2.The Volterra algebra $L^1[0,1]$[1] isn't topologically nilpotent ; For establishing this, consider $x_i(t) = \begin{cases} 2^i & 0 \leq t \leq 2^{-i} \\ \\ 0 & 2^{-i} \leq t \leq 1 \end{cases}$.
Then $\|x_i\| = 1$ $(i = 1, 2, \ldots)$ and for all n, $\|x_1 \ldots x_n\| = 1$.

[1]The Banach space $L^1([0,1])$ with the product $(fg)(x) = \int_0^x f(x-y)g(y)dy$ is a non-unital commutative Banach algebra and called Volterra algebra.

A finite dimensional commutative algebra with nilpotent radical, an identity modulo the radical, but no global identity.

****

Let $A = \mathcal{C}^2$ with multiplication $(a, b)(c, d) = (ac, 0)$ $(a, b, c, d \in \mathcal{C})$. Clearly $A^2 = A$. Its radical is $R = \{(0, b); b \in \mathcal{C}\}$ and $\frac{A}{R} \simeq \mathcal{C}$. The identity of $\frac{A}{R}$ lifts to the idempotent $(1, 0)$ in A [Ric, Theorem 2.3.9], but there is no identity in A.

Ref.

[**Ric**] C.E. Rickart, General theory of Banach algebras, Princeton, Van Nastrand, 1960.

A Banach algebra having no bounded approximate identity.

****

$\{xy\ ;\ x, y \in l^2\}$ is a proper subset of Banach algebra l^2 equipped with the coordinatewise operations. In fact $(\frac{1}{n}) \in l^2$ and if $x_n y_n = \frac{1}{n}$, then there exist an integer N such that for all $n > N$, $|x_n| \geq \frac{1}{\sqrt{n}}$ or for all $n > N$, $|y_n| \geq \frac{1}{\sqrt{n}}$, and hence $(x_n) \notin l^2$ or $(y_n) \notin l^2$. Now Cohen's factorization theorem [B&D,§11. Corollary 11] implies that l^2 has no bounded approximate identity.

Comment. Using BA37, we conclude that the Banach algebra l^2 has neither bounded aproximate identity nor unbounded one.

Ref.

[**B&D**] F.F. Bonsall and J. Duncan, Complete normed algebras, Springer-Verlag, 1973.

A Banach space with a non-complemented closed subspace.

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$c_\circ$ is a non-complemented closed subspace of l^∞.

(cf. [R.S. Philips, On linear transformations, Trans Amer Math. Soc. 48 (1940), 516-554.])

Newmann and Rudin gave another example, i.e. the subspace of $C(T)$ consisting of the boundary values of analytic functions.

(cf. [K. Hofman, Banach spaces of analytic functions, Prentice-Hall, Englewood Cliffs, N.J. 1962.])

A complete metrizable linear space whose metric cannot be obtained from a norm.

****

1. The linear space S consisting of all complex sequences with the metric $d((x_i),(y_i)) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1+|x_i - y_i|)}$ is a complete metric space. Since $d(2(1,1,\cdots),(0,0,\cdots)) \neq 2d((1,1,\cdots),(0,0,\cdots))$, the space (S,d) is not normable.

If $(X, \| \,.\, \|)$ is a normed linear space then $d(x,y) = \begin{cases} \| x-y \| + 1 & x \neq y \\ 0 & x = y \end{cases}$ is a metric on X, but can not be obtained from a norm, since if x is a nonzero vector of X then $d(2x,0) \neq 2d(x,0)$.

Comment. $C^{\infty}([0,1])$ with its usual topology is a complete metrizable linear space whose topology cannot be obtained from a norm.

Two non-isometrically isomorphic spaces with the same duals. So that a such dual space could not be a W*-algebra under any multiplication and involution.

****

c_0 and c are both closed subspaces of l^∞. In addition for each $x = (x_n) \in l^1$, $\rho_x : c_0 \longrightarrow \mathcal{C}$ given by $(y_n) \mapsto \sum_{n=1}^{\infty} x_n y_n$ is a bounded linear functional on c_0 with the norm $\| \rho_x \| = \| x \|$. Clearly $c_0^\#$ is isometrically isomorphic to l^1. Also for each $x = (x_n) \in l^1$, $\eta_x : c \longrightarrow \mathcal{C}$ given by $(y_n) \mapsto x_1 \lim_n x_n + \sum_{n=1}^{\infty} x_n y_n$ is a bounded linear functional on c with the norm $\| \eta_x \| = \| x \|$. Obviously $c^\#$ is isometrically isomorphic to l^1. But by BA25.DVI the closed unit ball of c_0 has no extreme point while the closed unit ball c contains at least $(1, 1, 1, \cdots)$ as an extreme point (since if $1 = tx_n + (1-t)y_n$ with $|x_n| \leq 1$ and $|y_n| \leq 1$, then $1 = tRex_n + (1-t)Rey_n$ for all n, so that $Rex_n = Rey_n = 1$ and hence $x_n = y_n = 1$ for each n). Thus c_0 and c_1 are not isometrically isomorphic.

Now by [Sak1, Corollary 1.13.3], l^1 can not be a W^*-algebra.

Re.

[**Sak1**] S. Sakai, C^*-algebras and W^*-algebras, Springer-Verlag, 1971.

A Banach space X such that all its closed subspaces are complemented.

****

Any Hilbert space.

Note that Lindenstrauss and Tzafriri, showed that each Banach space for which every closed subspace is complemented is isomorphic to a Hilbert space (cf. [J. Lindenstrauss and L. Tzafiriri, On complemented subspaces problem. Israel J. Math., 2, 1984, 375-378].)

A Banach space which isn't metrizable in weak topology.

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Every Hilbert space has this property (cf. [Hal, problem 21]).

Comment. It is probably true that no infinite dimensional Banach space is metrizable in the weak topology.

Ref.

[Hal] P.R. Halmos, A Hilbert space problem book, Princeton, Van Nostrand, 1967.

A Banach space which is not an inner product space.

****

The supremum norm on $C[a,b]$ can not be obtained from an inner product. Since if $f(t) = 1$ and $g(t) = \frac{t-a}{b-a}$, then $\| f \| = \| g \| = 1, \| f - g \| = \sup\{|1 - \frac{t-a}{b-a}|; t \in [a,b]\} = 1$ and $\| f + g \| = \sup\{|1 + \frac{t-a}{b-a}|; t \in [a,b]\} = 2$ and so the parallelogram equality $\| f + g \|^2 + \| f - g \|^2 = 2 \| f \|^2 + 2 \| g \|^2$ (which is satisfied in every inner product space) isn't held.

Comment. Indeed this Banach space is not an inner product space in any equivalent norm.

An incomplete inner product space.

****

The linear space $C[a,b]$ of all continuous complex-valued functions on $[a,b]$ with the inner product $< f,g >= \int_a^b f(x)\overline{g(x)}dx$ is not complete with respect to the norm $\| f \| =< f,f >^{\frac{1}{2}} = (\int_a^b |f(x)|^2 dx)^{\frac{1}{2}}$. In fact the sequence (f_n) where

$$f_n(x) = \begin{cases} 0 & a \le x < \frac{b+a}{2} \\ (n+n_0)(x - \frac{b+a}{2}) & \frac{b+a}{2} \le x \le \frac{b+a}{2} + \frac{1}{n+n_0} \\ 1 & \frac{b+a}{2} + \frac{1}{n+n_0} < x \le b \end{cases}$$

(n_0 is a natural number greater than $\frac{2}{b-a}$)
is a Cauchy but not convergent.

Two closed densely defined operators T and S on a Hilbert space such that $T+S$ isn't closable.

****

Consider a separable infinite dimensional Hilbert space H with an orthonormal basis (ξ_n). Let $D = \{\eta \in H; \sum_{n=1}^{\infty} n^4 | < \eta, \xi_n > |^2 < \infty\}, \zeta = \sum_{n=2}^{\infty} n^{-1}\xi_n$, and define the operators S and T with the domain D, which is dense in H, by

$$S\eta = \sum_{n=2}^{\infty} n^2 < \eta, \xi_n > \xi_n \quad , \quad T\eta = S\eta + < S\eta, \zeta > \xi_1 \quad (\eta \in D).$$

Then $-S$ and T are closed densely defined and $T + (-S)$ isn't closable. (cf. Problem 2.8.43 of [K&R1])

Ref

[K&R1] R.V. Kadilon and J.R. Ringrase, Fundamentals of the theory of operator algebras (I), Acad. Press, 1983.

A Hilbert space whose Hamel dimension and Hilbert dimension are different.

****

The Hilbert space l^2 has the orthonormal basis (e_n) with $e_n(m) = \delta_{mn}$; $m, n \in \mathbf{N}$. Hence its Hilbert dimension is $\aleph_0$. But the set of all sequences $x_\alpha =< 1, \alpha, \alpha^2, \alpha^3, \cdots >, 0 < \alpha < 1$ is a linearly independent uncountable subset of l^2. Thus the Hamel dimension of l^2 isn't $\aleph_0$.

Comment. This Hilbert dimension is probably the only one which this can happen.

A nonclosable unbounded operator on a Hilbert space.

****

Let H be a separable Hilbert space with the standard orthonormal basis (ξ_n). Define T on H by $T\xi_n = n\xi_1$ and extend T to the dense linear subspace $D(T)$ of finite linear combinations of basis elements ξ_n (we denote the extension of T by the same T). Then T is a densely defined unbounded operator on H (since $\lim_{n\to 0}\frac{\| T\xi_n \|}{\| \xi_n \|} = \lim_{n\to 0} n = \infty$). Moreover T is not closable, for $\lim_{n\to 0}\frac{\xi_n}{n} = 0$ but $\lim_{n\to 0} T(\frac{\xi_n}{n}) = \xi_1$.

On a separable infinite dimensional Banach space X there exists another norm under which X isn't separable.

****

Suppose that $\{e_i; i \in I\}$ is a Hamel basis for X and I is countable. For each $i \in I$, let X_i denote the linear span of $\{e_1, e_2, \cdots, e_n\}$, then $X = \cup_{i=1}^{n} X_i$. But the X_i are proper closed subspaces of X and so are nowhere dense, that is impossible by the Baire category theorem. Thus I is uncountable.

Let $a \in X, a = \sum\limits_{i \in I} \lambda_i e_i$ where all λ except finitely many are zero. Set $\| a \|' = \sum\limits_{i \in I} |\lambda_i|$. Then $\| \,.\, \|'$ is obviously a norm on X. For $i \neq j, \| e_i - e_j \|' = 2$ and I is uncountable, hence $(X, \| \,.\, \|')$ has no dense countable subset.

Notation

In this site we use $X^{\#}$ for the topological dual of a normed space$X$, S' for the commutant of a subset S of $B(H)$ and T^* for the Hilbert adjoint of an operator T in $B(H)$ for any Hilbert space H.

Main Examples

(I) The set of complex numbers $\mathcal{C}$ with usual addition, multiplication and the absolute value as a norm is a unital commutative Banach algebra.

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(II) $\mathcal{C}^n$ with the coordinatewise addition, scalar multiplication and the inner product

$$< (z_1, \ldots, z_n), (w_1, \ldots, w_n) >= \sum_{i=1}^{n} z_i \overline{w_i} \tag{1}$$

is a Hilbert space.

****

(III) The space $\mathcal{C}^2$ (see (II)) with the product $(a, b)(a', b') = (aa', ab'+a'b)$ is a unital commutative Banach algebra.

****

(IV) Let X be a non-empty set and Y is a normed (Banach) space. Then the set $l^\infty(X, Y)$ of all bounded mappings of X into Y with the pointwise addition $(f + g)(x) = f(x) + g(x), x \in X$; poinwise scalar multiplication

$(\lambda f)(x) = \lambda f(x), \lambda \in \mathcal{C}, x \in X$; and supremum norm $\|f\| = \sup\{|f(x)|; x \in X\}$ is a normed (Banach) space. If Y is normed algebra then $l^\infty(X, Y)$ with the pointwise product $(fg)(x) = f(x)g(x)$ is a normed algebra.
We denote $l^\infty(E, \mathcal{C})$ with $l^\infty(E)$ that is a unital cammutative C^*-algebra under the involution $f^* = \overline{f}$, the conjugate of f. Also $l^\infty(\mathcal{N})$ is denoted by l^∞.
The set of all convergent sequences of complex numbers, c, is a closed *-subalgebra of l^∞ and the set of all elements of c converging to zero, c_0, is a closed *-subalgebra of c.

****

(V) If X is a topological space, then the set $C_b(X)$ of all bounded continuous complex valued functions on X is a closed *-subalgebra of $l^\infty(X)$ containing the constant function 1. So $C_b(X)$ is a unital commutative C^*-algebra.

****

(VI) If X is a locally compact Hausdorff space, then the set $C_0(X)$ of all continuous complex valued functions on X vanishing at infinity (i.e. for each $\varepsilon > 0$, the set $\{x \in X; |f(x)| \geq \varepsilon\}$ is compact) is a closed *-subalgebra of $l^\infty(X)$ and so is a commutative C^*-algebra.
$C_0(X)$ is unital iff X is compact. Each non-unital commutative C^*-algebra is of this form (cf. [Mur]).

****

(VII) If X is a compact Hausdorff space, then the set $C(X)$ of all continuous complex functions on X is exactly $C_0(X)$ and so is a unital commutative C^*-algebra. Each unital commutative C^*-algebra is of this form (cf. [Mur]). By ([K&R1, Th. 5.3.1]), An abelian W^*-algebra is isometrically $*$-isomorphic to $C(X)$ for some extremely disconnected compact Hausdorff space X.
(A topological space is called extremely disconnected or Stonean if the closure of any open set is open).

****

(VIII) Let Δ denote the closed unit disc $\{z \in \mathcal{C}, |z| \leq 1\}$. Suppose that $A(\Delta)$ denoted the set of all elements of $C(\Delta)$ which are analytic on the interior of Δ. $A(\Delta)$ is a closed subalgebra of $C(\Delta)$ (Since if $f_n \in A(\Delta)$ and (f_n) converges to $f \in C(\Delta)$ in the norm of $C(\Delta)$ and γ is a simple closed path in the interior of Δ, then $\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz$ but by Cauchy's theorem $\int_\gamma f_n(z)dz = 0 (n \in \mathcal{N})$. So $\int_\gamma f(z)dz = 0$. Now Morera's theorem implies that f is analytic in the interior of Δ), and so it is a unital commutative Banach algebra. We call this the disc algebra.

****

(IX) Let (Ω, μ) be a measure space and $L^p(\Omega, \mu)$ for $1 \leq p < \infty$ be the set of all complex valued measurable functions f on Ω (we assume f is equal to g if $f = g$ a.e.$[\mu]$) for which $\|f\|_p = (\int_\Omega |f|^p d\mu)^{\frac{1}{2}} < \infty$. $L^p(\Omega, \mu)$ with the norm $\|.\|_p$ is a Banach space and is a Hilbert space iff $p = 2$. $L^p(\Omega, \mu)$ denoted by $l^p(\Omega)$ if μ is counting measure. In particular, $l^p(\mathcal{N})$ denoted by l^p. Let $H = l^2$, (α_n) be a bounded sequence of complex numbers, and (ξ_n)

be the (usual) standard orthonormal basis of H, that is, $(\xi_n)(m) = \delta_{nm}$, $n, m \in \mathcal{N}$ (δ denoted the kronecker delta), so that $\zeta = \sum_{n=1}^{\infty} < \zeta, \xi_n > \xi_n$ for any $\zeta \in H$. Then the operator $T \in B(H)$ defined by $T\xi_n = \alpha_n \xi_{n+1}$ is called a weighted shift with the weights (α_n). If $\alpha_n = 1$ for all n, then T is called unilateral shift operator. It is straightforward to show that $\|T\| = \sup_n |\alpha_n|$, $r(T) = \lim_k \sup_n |\prod_{i=0}^{k-1} \alpha_{n+i}|^{1/k}$ and $T^*\xi_1 = 0$ and $T^*\xi_n = \overline{\alpha_n}\xi_{n-1}$.
If $1 \leq p < \infty$, then l^p can be regarded as a commutative Banach algebra with coordinatewise multiplication. (For $p > 1$, $\|fg\|_p \leq \|f\|_p \|g\|_p$ is a conclusion of Hőlder inequality.) The l^p, $1 \leq p < \infty$, with the involution $f \longmapsto \overline{f}$ is an involutive Banach algebra.

****

(X) The Banach space $L^1([0,1])$ with the product $(fg)(x) = \int_0^x f(x-y)g(y)dy$ is a non-unital commutative Banach algebra. It is called Volterra algebra.

(XI) Let G be a locally compact group and μ a left invariant Haar measure on G, i.e. a Borel measure satisfying the following conditions.

(a) $\mu(xE) = \mu(E)$, for every $x \in E$ and every measurable $E \subseteq G$.

(b) $\mu(U) > 0$, for every non-void open set $U \subseteq G$.

(c) $\mu(K) < \infty$, for every compact set $K \subseteq G$.

With the notation IX, and under the product given by the convolution $(f * g)(s) = \int_G f(t)g(t^{-1}s)d\mu(t)$ $(s \in G)$, $L^1(G)$ is a commutative Banach algebra which called the group algebra of G. In particular, we can cansider $L^1(\mathcal{R})$, where the Lebesgue measure is an invariant Haar measure on $\mathcal{R}$. Also if G be an (algebraic) group, then G with the discrete topology is a locally compact group. A left invariant Haar measure on G is the counting measure on G. The corresponding group algebra, denoted by $l^1(G)$ and is called discrete group algebra.

****

(XII) Let S be a semi-group and α a positive real-valued function on S such that $\alpha(st) \leq \alpha(s)\alpha(t) \quad (s,t \in S)$. If $l^1(S,\alpha)$ is the set of all complex-valued functions f on S for which $\sum\limits_{s\in S}|f(s)||\alpha(s)| < \infty$, then $l^1(S,\alpha)$ with the usual pointwise addition and scalar multiplication and the product (convolution) $(f*g)(s) = \sum\limits_{tu=s} f(t)g(u)$ (if $tu = s$ has no solutions, we assume $(f*g)(s) = 0$), and with the norm $\|f\| = \sum\limits_{s\in S}|f(s)|\alpha(s)$is a Banach algebra. If $\alpha(s) = 1$, $l^1(S,\alpha) = l^1(S)$ is called discrete semi-group algebra, Moreover if $S = G$ is a group then $l^1(S)$ is the same discrete group algebra $l^1(G)$.

****

(XIII) Let (Ω,μ) be a measure space. Then the set $L^\infty(\Omega,\mu)$ consisting of all complex valued measurable functions f on Ω (with identifying functions which are almost everywhere equal) for which $\|f\|_\infty = \inf\{\lambda; \mu\{x \in \Omega; |f(x)| > \lambda\} = 0\} < \infty$ with the essential norm $\|.\|_\infty$ and pointwise operations is a unital commutative Banach algebra.

****

(XIV) If (Ω,μ) is a measure space, then $B_\infty(\Omega)$ that is the set of all bounded complex valued measurable functions on Ω is a closed subalgebra of $l^\infty(\Omega)$ and $L^\infty(\Omega,\mu)$ (again we identify almost everywhere equal functions).

****

(XV) The algebra $C^m([0,1])$ of the complex valued m times continuously differentiable on $[0,1]$ with the norm $\|f\| = \sum_{k=0}^{m} \frac{1}{K!} \sup_{x\in[0,1]} |f^{(k)}(x)|$ is a unital commutative Banach algebra. Its maximal ideals are precisely the $I_z = \{f; f(z) = 0\}$ where $z \in [0,1]$. Hence $C^m([0,1])$ is semi-simple.

****

(XVI) Suppose W is the set of all complex-valued functions f defined on the interval $[0, 2\pi]$ of the form $f(t) = \sum_{k\in\mathcal{Z}} \alpha_k \exp(ikt)$ $(t \in [0, 2\pi])$, where the $\alpha_k \in \mathcal{C}$ and $\sum_k |\alpha_k| < \infty$. The set W with the usual pointwise operations and with the norm $\|f\| = \sum_{k\in\mathcal{Z}} |\alpha_k|$ is a commutative Banach algebra and called the Wiener algebra. There is an isometric isomorphism between $l^1(\mathcal{Z})$ and W given by $f \longrightarrow \tilde{f}$ where $\tilde{f}(t) = \sum_{k\in\mathcal{Z}} f(k) \exp(ikt)$ $(t \in [0, 2\pi])$.

****

(XVII) Let X and Y are normed spaces. Then the set of all bounded linear mappings (bounded operators) from X into Y with the operator norm $\|T\| = \sup\{\|Tx\|; \|x\| \leq 1\}$ and with the pointwise addition and scalar multiplication is a normed space. It is Banach iff Y is Banach.
If $Y = X$, the space $B(X,X) = B(X)$ with the product $(ST)x = S(Tx)$ is a normed algebra (Banach algebra, if X is a Banach space).

****

(XVIII) In $(XVII)$ if $X = H$ is a Hilbert space, then $B(H)$ with the involution $T \longmapsto T^*$ being defined by $< T^*x, y >=< x, Ty >$ $(x, y \in H)$ is a C^*-algebra. Each C^*-algebra is isometrically isomorphic to a norm closed $*$-subalgebra of $B(H)$ for a Hilbert space H.

****

(XIX) An operator from normed space X into normed space Y is called compact if $T(U)$ is relatively compact in Y, where U is open unit ball of X; or equivalently for each bounded sequence (x_n) in $X, (Tx_n)$ has a convergent subsequent in Y. The set of all compact operators from X into Y is denoted by $K(X,Y)$ that is a subspace of $B(X,Y)$.

If X is a Banach space, $K(X) = K(X,X)$ is a closed two-sided ideal of $B(X)$.

****

(XX) Identifying $M_n(\mathcal{C})$, the algebra of all $n \times n$ matrices with entries in $\mathcal{C}$, with $B(\mathcal{C}^n) = K(\mathcal{C}^n)$. So it is a unital non-commutative $\mathcal{C}^*$-algebra.

****

(XXI) Let H be a Hilbert space and $x\overline{\otimes}y$ is the (one-rank) operator given by $(x\overline{\otimes}y)z =< z, y > x$. Suppose that $(e_i)_{i\in I}$, $(f_i)_{i\in I}$ are othonormal bases for H and $(\lambda_i)_{i\in I}$ is a family of complex numbers indexed by the same set I. The operator $T = \sum_{i\in I}\lambda_i e_i\overline{\otimes}f_i$ is well-defined and belongs to $B(H)$ iff (λ_i) is bounded and then $\|T\| = \sup\{|\lambda_i|; i \in I\}$.

****

(XXIa) An operator T is called of finite rank n if $n = dimT(H) < \infty$. The set $F(H)$ of all finite rank operators is a self-adjoint two-sided ideal of

$B(H)$. It is consisting of all operators as $\sum_{i\in I}\lambda_i e_i\overline{\otimes} f_i \in B(H)$ such that $\lambda_i = 0$ for all i except finitely many i.

****

(XXIb) The two-sided ideal of the compact operators $K(H)$ is self-adjoint and $F(H)$ is norm-dense in $K(H)$. $K(H)$ is consisting of all operators as $T = \sum_{i\in I}\lambda_i e_i\overline{\otimes} f_i \in B(H)$ such that the λ_i are positive (the λ_i^2 are the eigenvalues of T^*T). This sum has either a finite or a denumerably infinite number of terms; in the last case, $\lambda_i \to 0$.

****

(XXIc) The set $S(H)$ of all operators T for which $\sum_{i\in I}\|Te_i\|^2 < \infty$ is a self-adjoint ideal of $B(H)$. These operators are called Hilbert-Schmidt operators on H. The algebra $S(H)$ with the Hilbert-Schmidt norm $\|T\|_2 = (\sum_{i\in I}\|Te_i\|^2)^{1/2}$ is a Banach algebra. It contains operators of finite rank as a dense subset. For any pair of operators T and S in $S(H)$, the family $(< Te_i, Se_i >)_{i\in I}$ is summable. Its sum (A, B) defines an inner product in $S(H)$ and $(T,T)^{1/2} = \|T\|_2$. So $S(H)$ is a Hilbert space (independent on the choice basis (e_i)). $S(H) \subseteq K(H)$. $S(H)$ consists of precisely those compact operators $T = \sum_i \lambda_i e_i\overline{\otimes} f_i$ for which $\sum_i \lambda_i^2 < \infty$. In addition $\|T\|_2 = (\sum_i \lambda_i^2)^{1/2}$.

****

(XXId) The set of all products of two Hilbert-Schmidt operators is denoted by $N(H)$ and its elements are called trace-class operators. This set is a self-adjoint two-sided ideal of $B(H)$ and coincides with the set of those operators T for which $\sum_{i\in I} < |T|e_i, e_i >< \infty$ where $|T|$ is the absolute value of T in the C^*-algebra $B(H)$. If $\|T\|_1 = \sum_{i\in I} < |T|e_i, e_i >$, then $N(H)$ with this norm is a Banach algebra. $F(H)$ is a dense subset of $N(H)$. $N(H)$ is contained in $K(H)$ and contains $S(H)$. The elements of $N(H)$ are precisely the compact operators $T = \sum_{i\in I}\lambda_i e_i \overline{\otimes} f_i$ for which $\sum_i \lambda_i < \infty$. Moreover, $\|T\|_1 = \sum_i \lambda_i$.

****

(XXII) The set $\mathcal{C}[z]$ of all polynomials in an indeterminate z with complex coefficients under usual operations on polynomials and with the norm $\|p\| = \sup_{|\lambda|\leq 1} |p(\lambda)|$ is a normed algebra.

****

(XXIII) The set of all formal polynomials of degree at most n with the usual addition, scalar multiplication and product (but together with the convention that $x^k = 0$ if $k > n$) and with the norm $\|p\| = \sum_{k=1}^{n} |\alpha_k|$ $(p(x) = \sum_{k=1}^{n} \alpha_k x^k)$ is a finite dimensional Banach algebra.

****

(XXIV) The algebra $C([0,1])$ with the supremum norm $\|.\|$ and multiplication $(f * g)(t) = \int_0^t f(s)g(t-s)ds$ is a Banach algebra.

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A construction of a bounded approximate identity for a commutative C^*-algebra A.

****

Let $A = C_0(X)$ be a commutative C^*-algebra. Consider the set Λ consisting of all compact subsets of X. $(\Lambda, \subseteq)$ is a directed set. For each compact subset K of X, by Urysohn's lemma, there exists a function $f_K \in C_0(X)$ equal to 1 on K satisfying $0 \leq f \leq 1$. For each $g \in C_0(X)$ and given $\varepsilon > 0$, $K_0 = \{x \in X; |g(x)| \geq \varepsilon\}$ is compact. Hence for all $K \supseteq K_0$, $\|f_K g - g\|_\infty = \sup_{x \in X} |f_K(x)g(x) - g(x)| < \varepsilon$. Therefore $\lim_{K \in \Lambda} f_K g = g$, Thus $(f_K)_{K \in \Lambda}$ is a bounded aproximate identity for A.

Two element x, y in a C^*-algebra A such that $sp(xy) \neq sp(yx)$.

****

Let $A = B(l^2)$, x be the unilateral shift operator on l^2, defined by $T(\alpha_1, \alpha_2, ...) = (0, \alpha_1, \alpha_2, ...)$, and $y = T^*$. Then $TT^*(\alpha_1, \alpha_2, \ldots) = T(\alpha_2, \alpha_3, \ldots) = (0, \alpha_2, \alpha_3, \ldots)$ and $T^*T(\alpha_1, \alpha_2, \ldots) = T^*(0, \alpha_1, \alpha_2, \ldots) = (\alpha_1, \alpha_2, \ldots)$. Hence $sp(T^*T) = \{1\}$ but $0 \in sp(TT^*)$ (since $(TT^*)(1, 0, 0, \ldots) = (0, 0, \ldots)$).

An involutive Banach algebra A which isn't a C^*-algebra.

****

Consider $A = A(\Delta)$[1]. Then $f^*(z) = \overline{f(\bar{z})}$ gives an involution on A such that $\|f\| = \sup_{z \in D} |f(z)| = \sup_{z \in D} |f(\bar{z})| = \|f^*\|$. Consider $f(z) = z^2$ and $g(z) = z$, then g is self-adjoint and $f = gg^*$. So f is positive and we must have $sp(f) \subseteq [0, \infty)$ contradicting $sp(f) = \Delta$. Hence A isn't a C^*-algebra.

[1] Let Δ denote the closed unit disc $\{z \in \mathcal{C}, |z| \leq 1\}$. Suppose that $A(\Delta)$ denoted the set of all elements of $C(\Delta)$ which are analytic on the interior of Δ. $A(\Delta)$ is a closed subalgebra of $C(\Delta)$.

An involution # on Banach algebra $M_4(\mathcal{C})$, two normal matrix T and S such that $TS = ST$ but $TS^{\#} \neq S^{\#}T$, $S+T$ isn't normal and $\|SS^{\#}\| \neq \|S\|^2$.

****

Set

$$U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $Q^{\#} = U^{-1}Q^*U$ where Q^* denote the conjugate transpose of Q is an involution on $M_4(\mathcal{C})$. An straightforward computation shows that S and T has desired properties.

Ref.

[**Rud1**] W. Rudin, Functional analisis, McGraw-Hill, 1989.

A Banach algebra with a unique C^*-involution.

****

Every C^*-algebra has this property. Indeed if A is a untial Banach algebra which is C^*-algebra with respect to involutions $*$ and $\#$, then if $x = x^*$ and f be a state on A (i.e., by [K&R1, Theorem 4.3.2] is a bounded linear functional satisfying $\|f\| = f(1) = 1$) then, $f(x) = \overline{f(x^*)} = \overline{f(x)}$, so that $f(i(x - x^{\#})) = i(f(x) - \overline{f(x)}) = 0$. Therefore, by [K&R1, Proposition 4.3.3] $sp(i(x - x^{\#})) = \{0\}$. Hence $i(x - x^{\#}) = 0$, by [K&R1, Proposition 4.1.1.(i)]. So $x^{\#} = x = x^*$. For an arbitrary element x with the real and imaginary parts x_1 and x_2, we have $x^* = x_1 - ix_2 = x^{\#}$. (If A doesn't have a unit, it is enough to consider its unitization).

Ref.

[K&R1] R.V. Kadilon, J.R. Ringrase, fundamentals of the theory of operator algebras (I), Acad. Press, 1983.

A C^*-algebra in which invertible elements are dense.

****

Consider $l^\infty(\Omega)$, the C^*-algebra of all bounded mappings from a set Ω into $\mathcal{C}$, $f \in l^\infty(\Omega), \varepsilon > 0$. If

$$g(t) = \begin{cases} f(t) & |f(t)| \geq \varepsilon \\ \\ \varepsilon & |f(t)| < \varepsilon \end{cases}$$

we have $g \in l^\infty(\Omega)$, $\|g - f\| \leq 2\varepsilon$. Since $\inf |g(t)| \geq \varepsilon > 0$, g is invertible.

Comment. $C([a, b])$ provides a separable example.

A liminal C^*-algebra which isn't postliminal.

****

Let A denote Toeplitz algebra. $K(H^2)$ is liminal. $\frac{A}{K(H^2)}$ is $*$-isomorphic to $C(T)$, so it is abelian and therefore liminal. Hence A is postliminal .But identity representation of A on H^2 is irreducible and not finite dimensional, so A isn't liminal. For details see [Mur, Example 5.6.4].

Ref.

[**Mur**] G.J. Murphy, C^*-algebras and operator theory, Academic Press, 1990.

A closed subalgebra of a C^*-algebra that isn't self-adjoint.

****

The disc algebra $\mathcal{A}(\Delta)$ [1] is a closed subalgebra of the C^*-algebra $C(\Delta)$. If f and $\bar{f}$ both belong to $\mathcal{A}(\Delta)$, then by the Cauchy-Riemann equations f will be constant. So $\mathcal{A}(\Delta)$ isn't self-adjoint.

[1](VIII) Let Δ denote the closed unit disc $\{z \in \mathcal{C}, |z| \leq 1\}$. Suppose that $A(\Delta)$ denoted the set of all elements of $C(\Delta)$ which are analytic on the interior of Δ. $A(\Delta)$ is a closed subalgebra of $C(\Delta)$.

A closed left ideal of a C^*-algebra without any left approximate identity.

****

If ξ is a unit vector in a Hilbert space H with dimension at least 2, then $\Delta = \{T \in B(H); T\xi = 0\}$ is a closed left ideal in the C^*-algebra $B(H)$. If Δ has a left approximate identity $\{S_\alpha\}$ and $\eta \neq 0$ is a vector in H such that $< \xi, \eta >= 0$, then $\xi \overline{\otimes} \eta \in \Delta$ and so $\lim_\alpha S_\alpha(\xi \overline{\otimes} \eta) = \xi \overline{\otimes} \eta$. Thus $\lim_\alpha \|(S_\alpha \xi - \xi) \overline{\otimes} \eta\| = \lim_\alpha \|S_\alpha \xi - \xi\| \|\eta\| = 0$, hence $0 = \lim_\alpha \|S_\alpha \xi - \xi\| = \|\xi\|$, a contradiction. Thus Δ has no left approximate identity.

Note that for ζ_1 and ζ_2 in H the rank one operator $\zeta_1 \overline{\otimes} \zeta_2$ is defined by

$$(\zeta_1 \overline{\otimes} \zeta_2)(\zeta_3) =< \zeta_3, \zeta_2 > \zeta_1.$$

A nonclosed ideal that is not self-adjoint in a commutative C^*-algebra.

****

Consider C^*-algebra $A = C(\Delta)$ and the ideal $I = fA = \{fg\ ;\ g \in A\}$, where $f(z) = z$. $f^*(z) = \bar{z}$ and if $f^* \in I$, then there exists an element $g \in A$ such that $f^* = fg$. So $g(0) = \lim_{z\to 0} g(z) = \lim_{z\to 0} \frac{\bar{z}}{z}$, a contradiction. Thus I isn't self-adjoint.

A closed ideal I of a commutative C^*-algebra A and a closed ideal J of I such that J isn't an ideal of A.

****

Let $A = C([0,1])$, $I = Af$ and $J = \mathcal{C}f + Af^2$,where $f(t) = t; 0 \leq t \leq 1$. Then J is an ideal of I and I is an ideal of A; but $f \in J$ and $f.f^{\frac{1}{2}} \notin J$ (otherwise, there exist $\lambda \in \mathcal{C}$ and $g \in A$ such that $f.f^{\frac{1}{2}} = \lambda f + gf^2$. So $\lim_{t\to 0} t^{\frac{1}{2}} = \lambda + \lim_{t\to 0} tg(t)$. Therefore $\lambda = 0$ and $t^{\frac{1}{2}} = tg$ contradicting the continuity of g. Thus J isn't an ideal of A.

A C^*-algebra A where every unitary element is of the form $exp(ih)$ for a self-adjoint $h \in A$.

****

Suppose that $A = C([0,1])$. For each unitary $u \in A$, the mapping $t \longmapsto u_t$ from $[0,1]$ to the unitary group of G of A with $u_t(x) = u((1-t)x)$ connects u to $u(0)1$. If $u(0) = exp(i\theta)$ for some real number θ, $\{exp(it\theta)1; 0 \leq t \leq 1\}$ in G connects 1 to $u(0)1$. Therefore u is connected to 1. Now by [K&R3, Exercise 4.6.7], $u = exp(ih)$ for some $h \in A_h$.

Comment. By [K&R1, Theorem 5.2.1], A isn't W^*-algebra. **Ref.**

[K&R1] R.V. Kadilon, J.R. Ringrase, Fundamentals of the theory of operator algebras (I), Acad. Press, 1983.

[K&R3] R.V. Kadilon, J.R. Ringrase, Fundamentals of the theory of operator algebras (III), Acad. Press, 1991.

A C^*-algebra that isn't a von Neumann algebra.

****

$K(H)$, where H is a separable infinite dimensional Hilbert space is a C^*-algebra but not a von Neumann algebra. In fact if $(e_n)_{n \in \mathcal{N}}$ is a orthonormal basis for H and $P_n = \sum_{i=1}^{n} e_i \overline{\otimes} e_i$, then P_n is a finite-rank projection converging strongly to the identity operator I (since for each $x \in H$, $I(x) = x = \sum_{i=1}^{\infty} < x, e_i > e_i = \lim_n P_n(x)$). If $K(H)$ were a von-Neumann algebra, it should be $I \in K(H)$, a contradiction.

A C^*-algebra A in which the closed unit ball of A^+ isn't the closed convex hull of the projections of A. (Note that the closed unit ball of positive elements of each hereditary C^*-algebra A of a von Neumann algebra is the closed convex hull of its projections).

****

The only projections of $C([0,1])$ are 0 and 1. So the closed convex hull of $C([0,1])$ is $\{f \mid \exists c \in [0,1]; f = c\}$, not equal to $(C([0,1]))_1^+$.

A primitive C^*-algebra with a unique nontrivial closed bi-ideal (and so that it is not simple).

****

Let H be a separable infinite dimensional Hilbert space and $A = B(H)$. Then $K(H)$ is a nontrivial closed bi-ideal of $B(H)$, and if I is a nontrivial closed bi-ideal of $B(H)$, we have $F(H) \subseteq I$ (cf. [Mur, Th. 2.4.7]). Hence $K(H) \subseteq I$. If $I \not\subseteq K(H)$, then I has an infinite-rank projection p (cf. [Mur, Cor. 4.1.14]). For each infinite-rank projection q, there exist $u \in B(H)$ such that $p = u^*u$ and $q = uu^*$ (if (e_n) and (f_n) are orthonormal basis for $p(H)$ and $q(H)$ resp., define $u(e_n) = f_n$ and $u = 0$ on $p(H)^{\perp}$) so $q = upu^* \in I$. Hence $I = B(H)$, a contradiction.

Since $B(H)' = \mathcal{C}1$ (For $(\mathcal{C}1)' = B(H)$ and this is because of $(\mathcal{C}1)" = \mathcal{C}1$), the identity representation $B(H) \longrightarrow B(H)$ is a faithful irreducible representation. Hence $B(H)$ is primitive.

Ref.

[**Mur**] G.J. Murphy, C^*-algebras and operator theory, Academic Press, 1990.

A non-separable von Neumann algebra with a (unique) separable closed $*$-bi-ideal.

****

Let H be a separable infinite dimensional Hilbert space and (x_n) be a dense sequence in H. Then $K(H)$ which is the closed linear span of rank-one projections, is the closure of the linear span of $x_n \overline{\otimes} x_n$ with rational coefficients,hence it is separable. If (e_n) is an orthonormal basis for H and for each subset S of the natural numbers $\mathcal{N}$,

$$P_S(e_n) = \begin{cases} e_n & n \in S \\ \\ 0 & \text{otherwise} \end{cases},$$

then $\|P_S - P_{S'}\| = 1$, for $S \neq S'$. Thus $\{P_S\}_{S \in 2^N}$ cannot be in the closure of any countable sequence of $B(H)$. Thus $B(H)$ isn't separable.

Note that for x and y in H the rank one operator $x \overline{\otimes} y$ is defined by

$$(x \overline{\otimes} y)(z) = < z, y > x.$$

A primitive C^*-algebra A acting on a Hilbert space H such that $A \cap A' = \{0\}$ (A' is the commutant of A in $B(H)$).

****

Let H be an infinite dimensional Hilbert space, then $K(H)$ is primitive, since the identity representation

$$\begin{array}{ccc} K(H) & \longrightarrow & B(H) \\ T & \longmapsto & T \end{array}$$

is faithful irreducible (if $T \in K(H)'$, then for each x in H, $T(x\overline{\otimes}x) = (x\overline{\otimes}x)T$. So $Tx\overline{\otimes}x = x\overline{\otimes}T^*(x)$. Hence $< x, x > Tx =< x, T^*x > x$. So $Tx = \lambda(x)x$ for some $\lambda(x) \in \mathcal{C}$. For linearly independent vectors x and y, $\lambda(x+y)(x+y) = T(x+y) = Tx + Ty = \lambda(x)x + \lambda(y)y$. So $\lambda(x+y) = \lambda(x) = \lambda(y)$. Hence for each e in an orthonormal basis E of H, $\lambda(e) = \lambda(e_0)$, where e_0 is an arbitrary fixed element of E. Therefore $Tx = T(\sum_{e\in E} \mu_e e) = \sum_{e\in E} \mu_e \lambda(e_0)e = \lambda(e_0)x$. So $T = \lambda(e_0)I_H$. Thus $K(H)' \subseteq \mathcal{C}I_H$. Obviously $\mathcal{C}I_H \subseteq K(H)'$. So $K(H)' = \mathcal{C}I_H$). But $I_H \notin K(H)$. So $K(H) \cap (K(H))' = \{0\}$.

Note that for x and y in H the rank one operator $x\overline{\otimes}y$ is defined by

$$(x\overline{\otimes}y)(z) =< z, y > x.$$

A non-primitive C^*-algebra.

****

$C[0,1]$. In fact if A is a commutative primitive C^*-algebra, then A has a nonzero faithful irreducible representation (H,φ). So $(\varphi(A))' = \mathcal{C}1$. But $\varphi(A)$ is commutative, so $\varphi(A) \subseteq (\varphi(A))' = \mathcal{C}1$. But $\varphi(A) \neq \{0\}$ so $\varphi(A) = \mathcal{C}1$. Thus $A \simeq \varphi(A) = \mathcal{C}1$.

A simple C^*-algebra.

****

$K(H)$ is a simple C^*-algebra. For if I is a nonzero closed bi-ideal of $K(H)$, then it is a closed bi-ideal of $B(H)$, so by [Mur, Th. 2.4.7] $F(H) \subseteq I$, hence $K(H) = \overline{F(H)} \subseteq \bar{I} = I$. Therefore $I = K(H)$.

Ref.

[**Mur**] G.J. Murphy, C^*-algebras and operator theory, Academic Press, 1990.

A non-unital C^*-algebra with compact primitive ideal space.

****

If H is an infinite dimensional Hilbert space, then the non-unital C^*-algebra $K(H)$ is simple. By (CW17), $(K(H))' = \mathcal{C}1$, so the identity representation

$$\begin{aligned} K(H) &\longrightarrow B(H) \\ T &\longmapsto T \end{aligned}$$

is a faitful irreducible representation, hence $\{0\}$ is a primitive ideal of $K(H)$. By (CW19), $K(H)$ is simple, so primitive ideal space of $K(H)$ is $\{\{0\}\}$, a compact space.

A non-liminal (CCR) C^*-algebra.

****

Let H be an infinite dimensional Hilbert space. Then faithful irreducible representation

$$\begin{array}{ccc} B(H) & \longrightarrow & B(H) \qquad (B(H)' = \mathcal{C}1) \\ T & \longmapsto & T \end{array}$$

together with $B(H) \neq K(H)$, shows that $B(H)$ isn't liminal.

Comment. Another example may be found in CW22.

A C^*-algebra A and a closed bi-ideal J of A such that $\frac{A}{J}$ and J are liminal, but A is not liminal.

****

Let H be an infinite dimensional Hilbert space and I_H be the identity operator on H. Then $A = K(H) + \mathcal{C}I_H$ isn't liminal (otherwise, since identity representation $K(H) + \mathcal{C}I_H \longrightarrow B(H)$ is nonzero irreducible $((K(H) + \mathcal{C}I_H)' = K(H)' = \mathcal{C}I_H$ (see $CW17$)) we should have $I_H \in K(H)$, a contradiction). But $K(H)$ is liminal, since each nonzero irreducible representation of $K(H)$ is unitarily equivalent to identity representation $K(H) \longrightarrow B(H)$ (see [Mur, page 146]). Also $\frac{A}{J} \simeq \mathcal{C}I_H$ which is finite dimensional and so is liminal (Every finite dimensional C^*-algebra B is liminal, since if (H_1, Ψ) is a nonzero irreducible representation of B, then for $x \neq 0$ in H_1, $\Psi(B)x$ is finite dimensional (for $\Psi(B)$ is finite dimensional). If $(u_\lambda)_\lambda$ be any approximate unit for B, then $(\Psi(u_\lambda))_\lambda$ strongly converges to I_H so $x \in [\Psi(B)x] = \Psi(B)x$. Hence $\Psi(B)x$ is a nonzero (closed) subspace of H_1 invariant for $\Psi(B)$, so by irreducibility, $\Psi(B)x = H_1$. Therefore H_1 is finite dimensional. Thus $\Psi(B) \subseteq B(H_1) = K(H_1)$).

Ref.

[Mur] G.J. Murphy, C^*-algebras and operator theory, Academic Press, 1990.

An operator of index zero which isn't invertible.

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Let P be a non-trivial finite-rank idempotent in $B(X)$ (X is a Banach space), then $I - P$, the difference of an invertible operator and a compact operator, is Fredholm, of index $ind(I) = 0$, and non-invertible.

A compact operator with no eigenvalues.

****

Let $X = C([0,1])$, $v : X \longrightarrow X$ be the Volterra operator $v(f)(x) = \int_0^x f(t)dt$. If S is the closed unit ball of X, then $v(S)$ is equicontinuous and pointwise-bounded, hence by the Arzela-Ascoli theorem, v is compact. If for some $\lambda \in \mathcal{C}$ and $f \neq 0$ in X, $vf = \lambda f$, then $f(x) = \lambda f'(x)$. So $\lambda \neq 0$ and $lnf(x) = \frac{x}{\lambda} + c$ for some $c \in \mathcal{R}$. Hence $f(x) = f(0)e^{\frac{x}{\lambda}} = 0e^{\frac{x}{\lambda}} = 0$, a contradiction. v has then no eigenvalue.

A week-operator closed subalgebra B of bounded operators on a Hilbert space H such that $B \neq B^{"}$, where $B^{"}$ denotes the doubel commutant of B.

****

Let H be a Hilbert space of dimension greater than $1, \xi$ be a unit vector in H and B be the subalgebra of $B(H)$ consisting of those operators for which ξ is an eigenvector. Let P be the projection with range $[\xi]$ (If $K \subseteq H$, we denote the closed linear span of K by $[K]$). Then $T \in B$ iff $PTP = TP$. $B(H)$ with weak-operator topology is Hausdorff and the mappings $T \longrightarrow PTP$ and $T \longrightarrow TP$ are weak-operator continuous, hence B is weak-operator closed in $B(H)$.

Choose a unit vector $\eta \in H$ orthogonal to ξ. Suppose that Q is the projection onto $[\{\xi, \eta\}]$ and S is the operator defined by $S\eta = \xi, S\xi = 0$ and $S(I-Q) = 0$. Then P, Q and S are in B. Thus if $T' \in B'($ the commutant of $B)$, then ξ and η are eigenvectors for T', say $T'\xi = \alpha\xi$ and $T'\eta = \beta\eta$. Since $T'S = ST', \beta\xi = \beta S\eta = ST'\eta = T'\xi = \alpha\xi$ and $\alpha = \beta$. But η is an arbitrary element orthogonal to ξ; therefore $T' = \alpha I$. Thus $B' = \{\alpha I; \alpha \in \mathcal{C}\}$. (Here I denotes the identity operator on H.)

A unitary operator U acting on a Hilbert space whose spectrum is $C = \{z \in \mathcal{C}; |z| = 1\}$.

****

If H is a separable infinite dimensional Hilbert space with an orthogonal basis $(\xi_n)_{n\in\mathcal{Z}}$, we define $U\xi_n = \xi_{n+1}$. Then U is isometric and surjective, so it is a unitary operator. By Lemma 3.2.13 of [K&R1], $sp(U) \subseteq C$. If $\lambda \in C$ and $x_n = (2n+1)^{\frac{-1}{2}} \sum_{k=-n}^{n} \lambda^{-k}\xi_k$, then $\| \xi_n \|= 1$ and $\| (U - \lambda I)x_n \|= (2n+1)^{\frac{-1}{2}} \| \sum_{k=-n}^{n} \lambda^{-k}\xi_{k+1} - \sum_{k=-n}^{n} \lambda^{-(k-1)}\xi_k \|= (2n+1)^{\frac{-1}{2}} \| \lambda^{-n}\xi_{n+1} - \lambda^{n+1}\xi_{-n} \|= 2^{\frac{1}{2}}(2n+1)^{\frac{-1}{2}} \to 0$.

Therefore by the same lemma, $\lambda \in sp(U)$. Thus $sp(U) = C$.

Ref.

[K&R1] R.V. Kadison and J.R. Ringrose, Fundamental of the theory of Operator Algebras (I), Acad. Press, 1983.

An unbounded symmetric operator on an inner product space.

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Suppose that H is the subspace of l^2 consisting of all sequences (ζ_n) with $\zeta_n = 0$ for all sufficiently larg n. H is not complete (Since (a_n) where $a_n = (1, \frac{1}{2}, \cdots, \frac{1}{n}, 0, 0, \cdots)_{n\in\mathcal{N}}$ is a Cauchy divergent sequence in H).
Let T denote the linear mapping $(\zeta_n) \mapsto (n\zeta_n)$ on H. T is symmetric, for $< T((\zeta_n)), (\eta_n) >= \sum_{n=1}^{\infty} n\zeta_n\overline{\eta_n} =< (\zeta_n), T((\eta_n)) >$. T is unbounded since if (ξ_n) is the orthonormal basis for l^2, for each $n, \xi_n \in H, \| \xi_n \|= 1$ and $\| T\xi_n \|= n$.

Two selfadjoint operators T and S on a Hilbert space such that $sp(ST)$ is not a subset of $\mathcal{R}$.

****

Consider

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

belonging to $B(\mathcal{C}^2)$. Then S and T are Hermetian, but $sp(ST) = \{i, -i\}$ which is not a subset of $\mathcal{R}$.

Two Hermetian operators T and S on a Hilbert space such that $S \geq 0$ and $-S \leq T \leq S$ but not $|T| \leq S$.

****

Let

$$S = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sqrt{2} & \sqrt{2} \end{pmatrix} \geq 0, T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(S and T belong to $B(\mathcal{C}^2)$.)

Then

$$S - T = \begin{pmatrix} \sqrt{2} & 0 \\ \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 1 \end{pmatrix} \geq 0, S + T = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \geq 0$$

and so $-S \leq T \leq S$, but $S - |T| = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$ and $< (S - |T|)\xi, \xi >= -1$ where $\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, hence S dosen't majorize $|T|$.

A selfadjoint operator $T \neq 0$ on a Hilbert space such that T is neither positive nor negative.

****

Consider $T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ belonging to $B(\mathcal{C}^2)$, $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\eta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $< T\xi, \xi >= -1$ and $< T\eta, \eta >= 1$. Hence the selfadjoint operator T is neither positive nor negative.

A bounded operator on a Hilbert space which has no square root.

****

Suppose that T is the operator $T(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$ on l^2 (in fact T is the adjoint of the unilateral shift operator). If T has a square root S, then $S^2 = T$ and $KerS \subseteq KerT = \mathcal{C}\xi_1$ in which $\xi_1 = (1, 0, 0, \cdots)$. Since T is not one to one we conclude that S is not one to one. So that $KerS = \mathcal{C}\xi_1$. T is surjective, hence S is onto. So there exists an element η such that $S\eta = \xi_1$. Since $T\eta = S^2\eta = S\xi_1 = 0$, we have $\eta = \lambda\xi_1$ for some $\lambda \in \mathcal{C}$ and hence $\xi_1 = S\eta = \lambda S\xi_1 = 0$, a contradiction.

Comment. There is an open subset of $L(H)$ consisting of invertible operators with no square roots.

Ref.

J.B. Conway and B.B. Morrel, Roots and logarithms of bounded operators on Hilbert spaces, J. Funct Anal, 70(1987), 171-193.

A bounded increasing sequence of self-adjoint operators on a Hilbert space which is not uniformly convergent.

****

Assuming $(\xi_n)_{n\in\mathcal{N}}$ as an orthonormal basis for a separable infinite dimensional Hilbert space H, say l^2. Denote the linear span of $\{\xi_1, \xi_2, \cdots, \xi_n\}$ by Y_n. Let P_n be the projection onto the closed subspace Y_n. If $m < n$, then $Y_m \subset Y_n$ and so $0 < P_m < P_n$. Moreover $P_n - P_m$ is a projection and so $\| P_n - P_m \| = 1$ whenever $n \neq m$. Therefore (P_n) is an increasing sequence of self-adjoint operators which is not even a Cauchy sequence in uniform topology on $B(H)$.

Given a compact subset K of $\mathcal{C}$, there exists a bounded operator T on a Hilbert space such that $sp(T) = K$ and the set of eigenvalues of T is dense in K.

****

Suppose that $H = l^2$, (e_n) is the standard orthonormal basis for H and (λ_n) is a dense sequence in K. Set $T(\sum_{n=1}^{\infty} \alpha_n e_n) = \sum_{n=1}^{\infty} \lambda_n \alpha_n e_n$ where $(\alpha_n) \in l^2$. Obviously $K \subseteq sp(T)$. If $\lambda \notin K$, then $\inf\{|\lambda - \mu|; \mu \in K\} > 0$ and so $S(\sum_{n=1}^{\infty} \alpha_n e_n) = \sum_{n=1}^{\infty} (\lambda - \lambda_n)^{-1} \alpha_n e_n$ is a well-defined operator on H. S is the inverse of $\lambda I - T$. Therefore $\lambda \notin sp(T)$. Thus $K = sp(T)$.

For every n, $Te_n = \lambda_n e_n$. In fact $\{\lambda_1, \lambda_2, \cdots\}$ is the set of all eigenvalues of T that is dense in $sp(T)$.

Operators of arbitrary large norms that are bounded by 1 on a given basis of a separable infinite dimensional Hilbert space H.

****

Let (ξ_n) be an orthonormal basis for H. For $k \in \mathcal{N}$, define T_k on H by $T_k\eta =< \eta, \xi_1 + \xi_2 + \cdots + \xi_k > \xi_1$. Then

$$T_k\xi_n = \begin{cases} \xi_1 & n \leq k \\ 0 & n > k \end{cases}$$

Hence $\| T_k\xi_n \| \leq 1 (n \in \mathcal{N})$. On the other hand $T_k^*\eta =< \eta, \xi_1 > (\xi_1 + \cdots + \xi_k) \quad (\eta \in H)$; therefore $\| T_k \| = \| T_k^* \| \geq \| T_k^*\xi_1 \| = \| \xi_1 + \cdots + \xi_k \| = \sqrt{k}$.

Given a compact subset K of $\mathcal{C}$ such that $\overline{K^0} = K$, there exists an operator T acting on a Hilbert space H such that $sp(T) = K$ and T has no eigenvalue.

****

Let $H = L^2(K)$ in which K is equipped with the Lebesgue measure m on $\mathcal{R}^2$. Define T on H as the following:

$$(Tf)(\mu) = \mu f(\mu);\ \ \mu \in K, f \in H.$$

If $\lambda \notin K$, then $\sup\{|\lambda - \mu|^{-1}; \mu \in K\} < \infty$ and so we can define an operator S on H by $(Sf)(\mu) = (\lambda - \mu)^{-1} f(\mu); f \in H, \mu \in K$. Hence $S(T - \lambda I) = (T - \lambda I)S = I$ so that $\lambda \notin sp(T)$. If $\lambda \in K, (\lambda I - T)^{-1} \in B(H)$ and f denotes the characteristic function of $\{\mu; |\lambda - \mu| < \epsilon\}$ multiplied by $m(\{\mu; |\lambda - \mu| < \epsilon\})^{-1/2}$, then

$$\begin{aligned} 1 &= \| f \|_2 \leq \| (\lambda I - T)^{-1} \| \| (\lambda I - T) f \|_2 \\ &= \| (\lambda I - T)^{-1} \| \int_K (\lambda - \mu) f(\mu) dm(\mu) \leq \| (\lambda I - T)^{-1} \| \epsilon, \end{aligned}$$

a contradiction. Hence $(\lambda I - T)$ is not invertible. So $\lambda \in sp(T)$. It follows that $sp(T) = K$. In addition, if $Tf = \alpha f$ for some $\alpha \in \mathcal{C}$, then for all $\mu \in K$, $\mu f(\mu) = \alpha f(\mu)$. So $f = 0$ almost every where. Thus T has no eigenvalue.

An operator T on a Hilbert space such that the set $eig(T)$ of all eigenvalues of T is empty but $sp(T) \neq \emptyset$.

****

The unilateral shift operator on the Hilbert space l^2 (with its standard orthonormal basis (e_n)) given by $Te_n = e_{n+1}, n \in \mathcal{N}$, has no eigenvalue; since obviously $0 \notin eig(T)$ and if $0 \neq \lambda \in eig(T)$ and $Tx = \lambda x$ for some $x = \sum_{n=1}^{\infty} \alpha_n e_n \neq 0$, then $\sum_{n=1}^{\infty} \alpha_n e_{n+1} = \sum_{n=1}^{\infty} \lambda \alpha_n e_n$ and hence $\alpha_n = 0$ for all n, i.e. $x = 0$, a contradiction.

Next observe that $0 \in sp(T)$; otherwise T would be invertible so $T(T^{-1}(e_1)) = e_1$, but $< T(T^{-1}e_1), e_1 >= 0$ by the definition of T, that is impossible.

A Hilbert space H such that on $B(H)$

(i) the involution isn't continuous with respect to the strong operator topology;

(ii) the weak operator topology and the strong operator topology are different;

(iii) the operator norm is not continuous with respect to the strong operator topology and the weak operator topology;

(iv) the weak operator topology and the strong operator topology aren't metrizable;

(v) the operation multiplication is continuous in neither weak nor strong operator topology.

****

Let $H = l^2$ and (e_n) be the standard orthonormal basis for H (note that for all $x \in H, x = \sum_{n=1}^{\infty} < x, e_n > e_n$). Set $T_n = e_1 \overline{\otimes} e_n$. Then $T_n^* = e_n \overline{\otimes} e_1$

(i) $\lim_n \parallel T_n x \parallel = \lim_n \parallel < x, e_n > e_1 \parallel = \lim_n \mid < x, e_n > \mid = 0$. So $T_n \longrightarrow 0$ in the strong operator topology. But $\lim_n \parallel T_n^* e_1 \parallel = \lim_n \parallel e_n \parallel = 1$, hence T_n^* dosen't converge to zero in the strong operator topology. So $T \longrightarrow T^*$ is not continuous in the strong operator topology.

(ii) The involution is continuous with respect to the weak operator topology (since $\mid < Tx, y > \mid = \mid < T^* y, x > \mid$). Hence (i) implies that the weak operator topology and the strong operator topology don't coincide on $B(H)$.

(iii) $\parallel T_n \parallel = \parallel e_1 \parallel \parallel e_n \parallel = 1$, and by (i) $T_n \longrightarrow 0$ in the strong operator topology. Therefore the operator norm is not continuous on $B(H)$.

(iv) Let $\Delta = \{ n^{\frac{1}{2}} T_n ; n \in \mathcal{N} \}$. For each neighborhood $U(0, x_1, \cdots, x_m, \epsilon)$ of 0 in the strong operator topology, with $x_k = \sum_{n=1}^{\infty} \alpha_k^n e_n$, $\| n^{\frac{1}{2}} T_n x_k \| = n^{\frac{1}{2}} |\alpha_k^n|$.

But for every $1 \leq k \leq m$, $\sum_{n=1}^{\infty} |\alpha_k^n|^2 < \infty$, hence for each ϵ there exists a natural number n such that $n^{\frac{1}{2}} |\alpha_k^n| < \epsilon$. So that 0 belongs to the strong closure of Δ. It follows from the principle of uniform boundedness and $\| \sqrt{n} T_n \| = \sqrt{n}$ that any sequence in Δ doesn't converge to 0 in the strong operator topology. Hence the strong operator is not metrizable. Similarly one can show that the weak operator topology is not metrizable.

(v) Let Λ be the set of all (n, U) in which $n \in \mathcal{N}$ and U is a neighborhood of 0 in the strong operator topology on $B(H)$. Then Λ with the following relation is a directed set:

$$(m, U) \leq (m', U') \Leftrightarrow (m \leq m' \text{ and } U \supseteq U')$$

Suppose that S is the unilateral shift operator on (e_n), i.e. $S(\sum_{k=1}^{\infty} \alpha_k e_k) = \sum_{k=1}^{\infty} \alpha_k e_{k+1}$.

Obviously $S^*(\sum_{k=1}^{\infty} \alpha_k e_k) = \sum_{k=1}^{\infty} \alpha_{k+1} e_k$. If $\lambda = (m_\lambda, U_\lambda) \in \Lambda, \lim_n \| S^{n^*} x \| = m_\lambda \lim_n (\sum_{k=1}^{\infty} |\alpha_{k+n}|^2)^{\frac{1}{2}} \longrightarrow 0$ whenever $x = \sum_{k=1}^{\infty} \alpha_k e_k \in H$. Therefore $(m_\lambda S^{n^*})_{n \in \mathcal{N}}$ converges to 0 in the strong operator topology. So that there exists a positive integer number n_λ such that $m_\lambda S^{n_\lambda{}^*} \in U_\lambda$. Set $T_\lambda = m_\lambda S^{n_\lambda{}^*}$ and $R_\lambda = \frac{1}{m_\lambda} S^{n_\lambda}$. Then $\lim_\lambda \| R_\lambda \| = \lim_\lambda \frac{1}{m_\lambda} = 0$, so that R_λ converges to 0 in the norm topology.

If U be a strong neighborhood of 0 and $\lambda_0 = (1, U)$, then $T_{\lambda_0} \in U_{\lambda_0}$ and for every $\lambda \geq \lambda_0, T_\lambda \in U_\lambda \subseteq U_{\lambda_0}$. therefore $(T_\lambda)_{\lambda \in \Lambda}$ converges to 0 in the strong operator topology. But $T_\lambda R_\lambda = 1$ for all λ, hence if the multiplication is jointly continuous in either the weak or the strong operator topology, then $1 = \lim_\lambda T_\lambda R_\lambda = \lim_\lambda T_\lambda \lim_\lambda R_\lambda = 0$, a contradiction.

Comment. The statements are true on any infinite dimensional Hilbert space.

Ref.

[**Mur**] G.J. Murphy, C^*-algebras and operator theory, Academic Press, 1990.

A sequence of nilpotent operators on H which converges with respect to the norm topology on $B(H)$ to an operator which is not topologically nilpotent.

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This example is due to Kakutani (cf. [Ric, p. 282]). Let H be a separable Hilbert space with orthonormal basis $(f_m)_{m\in\mathcal{N}}$. Define $\alpha_m = e^{-k}$ for $m = 2^k(2l+1),\ k,l = 0,1,\cdots$ and also the operator T by $Tf_m = \alpha_m f_{m+1}, m \in \mathcal{N}$. Then $\| T \| = \sup_{m\in\mathcal{N}} |\alpha_m|, T^n f_m = \alpha_m\alpha_{m+1}\cdots\alpha_{m+n-1}f_{m+n}$ and so $\| T^n \| = \sup_{m\in\mathcal{N}}(\alpha_m\alpha_{m+1}\cdots\alpha_{m+n-1})$.

Moreover, by the definition of the α_m, we have $\alpha_1\alpha_2\cdots\alpha_{2^t-1} = \prod_{j=1}^{t-1} exp(-j2^{t-j-1})$.

Therefore $(\alpha_1\alpha_2\cdots\alpha_{2^t-1})^{\frac{1}{2^{t-1}}} > (\prod_{j=1}^{t-1} exp[\frac{-j}{2^{j+1}}])^2$ and if $\sigma = \sum_{j=1}^{\infty}\frac{j}{2^{j+1}}$, then $e^{-2\sigma} \leq \lim_n \| T^n \|^{\frac{1}{n}}$. So T is not topologically nilpotent.

Next define the operator T_k by

$$T_k f_m = \begin{cases} 0 & m = 2^k(2l+1), l = 0,1,\cdots \\ \alpha_m f_{m+1} & \text{otherwise} \end{cases}$$

Then T_k is nilpotent. But

$$(T - T_k)f_m = \begin{cases} e^{-k}f_{m+1} & m = 2^k(2l+1), l = 0,1,\cdots \\ 0 & \text{otherwise} \end{cases}$$

Thus $\| T_k - T \| = e^{-k}$, hence $\lim_k T_k = T$ in the norm topology on $B(H)$.

Ref.

[**Ric**] C.E. Rickart, General theory of Banach algebras, Princeton, Van Nastrand, 1960.

(a) A Banach space X and an operator $T \in B(X)$ having no nontrivial invariant subspace.
(b) A Banach space X and an operator $T \in B(X)$ having a nontrivial invariant subspace.

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(a) C.J. Read showed that if $X = l^1$ then there exists a bounded operator on l^1 having no nontrivial invariant subspace.

(cf. [C.J. Read, A solution to the invariant subspace problem, Bull. London Math. Soc., 16(1984), 337-401.]

(b) If $X = \mathcal{C}^n (n > 1), T \in B(\mathcal{C}^n) - \mathcal{C}I$ is an arbitrary operator and $\alpha \in \mathcal{C}$ is an eigenvalue of T, then $M = Ker(T - \alpha I)$ is a nontrivial subspace of X and $TM \subseteq M$.(I is the identity operator on $\mathcal{C}^n$)

(a) An injective operator on a Hilbert space H such that the range of $T, R(T)$, isn't dense in H.
(b) An operator S such that S is surjective but $Ker(S) \neq \{0\}$.

****

Let H be a separable Hilbert space with the standard orthonormal basis (e_n).
(a) The unilateral shift operator $T(\alpha_1, \alpha_2, \cdots) = (0, \alpha_1, \alpha_2, \cdots)$ on H is injective and the closure of its range is the closed linear span $\{e_2, e_3, \cdots\}$ which doesn't contain e_1.

(b) If $S = T^*$, then $S(\alpha_1, \alpha_2, \cdots) = (\alpha_2, \alpha_3, \cdots)$. So S is surjective but $Ker(S) \neq \{0\}$, since it is the linear span of e_1.

Two positive operators $T \leq S$ acting on a Hilbert sace such that S^2 does not majorize T^2.

****

Define T and S as operators on $\mathcal{C}^2$ by $T(z_1, z_2) = (z_1, 0)$ and $S(z_1, z_2) = (2z_1 + z_2, z_1 + z_2)$.

Then

$$sp(T) = sp(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) = \{0, 1\} \subseteq \mathcal{R}^{\geq 0}, T^* = T, sp(S) = sp(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}) = \{\frac{3 \pm \sqrt{5}}{2}\} \subseteq \mathcal{R}^{\geq 0},$$

$$S^* = S, sp(S - T) = sp(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}) = \{0, 2\} \subseteq \mathcal{R}^{\geq 0}.$$

Hence $0 \leq T \leq S$. But $sp(S^2 - T^2) = sp(\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}) = \{3 \pm \sqrt{10}\}$ is not a subset of $\mathcal{R}^{\geq 0}$. Therefore S^2 doesn't majorize T^2.

An unbounded operator on a Hilbert space H annihilating an orthonormal basis of H.

****

Let (e_n) be the standard orthonormal basis for the Hilbert space $H = l^2$. Extend (e_n) to a Hamel basis β for l^2. Choose $f \in \beta$ distinct to the e_n and define then the linear mapping $T : H \longrightarrow H$ by

$$T(g) = \begin{cases} 1 & g = f \\ 0 & g \in \beta \setminus \{f\} \end{cases}$$

Then $T(e_n) = 0$ and T is unbounded (otherwise, $1 = T(f) = \sum_{n=1}^{\infty} < f, e_n > Te_n = 0$).

An operator U on a Hilbert space, other than I, such that $sp(U) = \{1\}$ and $\| U \| = 1$.

****

Suppose that $H = L^2(0,1)$ with respect to the Lebesgue measure and $(Tf)(x) = \int_0^x f(t)dt$. It follows from BA15.DVI, $sp(T) = 0$, so that $sp(I + T) = \{1\}$. Hence $U = (I+T)^{-1} \neq I$ is well-defined, moreover $sp(U) = \{\lambda^{-1}; \lambda \in sp(I+T)\} = \{1\}$. Therefore

$$1 = r(U) \leq \| U \| .$$

But $\| U \| \leq 1$, since

$$\| U^{-1}\xi \|^2 = \| (I+T)\xi \|^2 = \| f \|^2 + < (T+T^*)\xi, \xi > + \| T\xi \|^2 \geq \| f \|^2 .$$

(Note that $T + T^*$ is a projection onto the space of constant functions, since $(T^*f)(t) = \int_t^1 f(t)dt$.)

Thus $\| U \| = 1$.

A unital commutative Banach algebra with a maximal ideal M of codimension 1 and a Banach A-module X such that $H^2(A,X) = 0$ but $H^2(M,X) \neq 0$.

****

Let $A = \mathcal{C}^2$ with the product $(a,b)(a',b') = (aa', ab' + a'b)$. $M = \{0\} \oplus \mathcal{C}$, being the kernel of the character $\phi : A \longrightarrow \mathcal{C}$ defined by $\phi(z,w) = z$, is a maximal ideal of codimension 1. Regard $X = \mathcal{C}$ as an annihilator A-module. By [B&D&L, Proposition 2.2],

$H^2(A,X) = \{0\}$. If $\mu((0,w_1),(0,w_2)) = w_1 w_2$, then $\mu \in Z^2(M,X)$, but $\mu \notin N^2(M,X)$ (otherwise $w_1 w_2 = \mu((0,w_1),(0,w_2)) = (\delta^1\lambda)((0,w_1),(0,w_2)) = (0,w_1).\lambda((0,w_2)) - \lambda((0,w_1).(0,w_2)) + \lambda(0,w_1).(0,w_2) = 0 - \lambda(0) + 0 = 0$, for all $w_1, w_2 \in \mathcal{C}$, a contradiction).

Thus $H^2(M,X) \neq 0$.

Ref.

[**B&D&L**] W.G. Bode, H.G. Dales and Z. Lykova, Algebraic and strany splittings of extensions of Banach algebras, mem. Amer. Math. Soc. 137 (1999).

A non-split short complex of Banach spaces whose dual splits.

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$0 \longrightarrow c_0 \xrightarrow{i} l^\infty \xrightarrow{\pi} \frac{l^\infty}{c_0} \longrightarrow 0$ is a short exact complex of Banach spaces which doesn't split since c_0 is not complemented in l^∞.
Its dual complex $0 \longrightarrow (\frac{l^\infty}{c_0})^\# \xrightarrow{\pi^\#} (l^\infty)^\# = (l^1)^{\#\#} \xrightarrow{i^\#} c_0^\# = l^1 \longrightarrow 0$ splits.
Notice that the later complex is exact and the canonical embedding $l^1 \longrightarrow (l^1)^{\#\#}$ is a right inverse to $i^\#$.

A weakly amenable commutative Banach algebra which is not amenable.

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$l^p, 1 \leq p < \infty$, is closed linear span of its idempotents $\zeta_n(k) = \delta_{nk}, n, k \in \mathcal{N}$. Suppose that e is an idempotent of l^p, X is a symmetric Banach l^p-module , and $D : A \longrightarrow X$ is a continuous derivation. Then $De = D(e^2) = 2eD(e)$ and so $De = (De - 2eDe) - 2e(De - 2eDe) = 0$. It follows that $D = 0$. So that l^p is weakly amenable. But $l^p, 1 \leq p < \infty$ has no bounded approximate identity (see ba41.dvi in the case $p = 2$), so that A is not amenable (cf. [B.E. Johnson, Cohomology in Banach algebras, Mem. Amer. Math. Soc. 127 (1972)]).

A derivation on an algebra which is not inner.

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Suppose that A is an algebra with unit 1 and a is an element of A which is not algebraic (i.e. $\{1, a, a^2, \cdots\}$ isn't a linearly independent set). Let B be the subalgebra of A generated by 1 and a. Define a mapping D of B into B by $D(\lambda_0 + \lambda_1 a + \cdots + \lambda_n a^n) = \lambda_1 + 2\lambda_2 a + \cdots + n\lambda_n a^{n-1}$. Obviously D is a derivation on B and isn't inner, since B is commutative and $D \neq 0$.

A closed unbounded $*$-derivation on a C^*-algebra A.

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Suppose that $A = C([0,1])$ and $\delta(f) = (\frac{d}{dt})f(t) = f'(t)$ with the domain $D(\delta) = C^1([0,1])$ where $C^1([0,1])$ is the algebra of all continuously differentiable functions on $[0,1]$. $\| \delta(x^n) \| = n = n \| x^n \|$ implies that δ is an unbounded derivation from $D(\delta)$ into A. If $f_n \in D(\delta), f_n \longrightarrow f \in A$, and $\delta(f_n) \longrightarrow g$, then $f_n' \longrightarrow g$ uniformly on $[0,1]$ and $f_n(0) \longrightarrow f(0)$. So g is differentiable and $f' = g$. Therefore $f \in D(\delta)$ and $\delta(f) = g$.

A Banach algebra for which every linear operator is a derivation.

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Suppose that A is an arbitrary Banach space. Defining $x.y = 0(x, y \in E), E$ is a Banach algebra. Obviously every linear operator is a derivation.

A non-closable unbounded $*$-derivation.

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Let X be the Cantor set of $[0,1]$. It is well-known that X is a perfect compact subset of $[0,1]$. By Tietze's theorem, $C(X) = \{f|_X; f \in C([0,1])\}$. Define δ on $D(\delta) = \{f|_X; f \in C^1([0,1])\}$ by $\delta(f|_X) = f'|_X$. δ is a well-defined derivation (if $f|_X = 0$, then for each $x_0 \in X$ there exists a sequence $\{x_n\}$ in $X - \{x_0\}$ converging to x_0. So $f'(x_0) = \lim_n \dfrac{f(x_n) - f(x_0)}{x_n - x_0} = 0$, therefore $f'|_X = 0$).

But δ is not identically zero so by [Sak2, Proposition 3.2.1] δ cannot be extended to a closed derivation in $C(X)$. So that δ isn't closable.

This example is due to O.Bratteli and D.W. Robinson ([cf. Sak2, P.59]).

Ref.

[**Sak2**] S. Sakai, Operator algebras in dynamical systems, Cambridge Univ. Press, 1991.