partem altitudinis multiplicetur, ostendam, quem ad modum, si latera pyramidis fuerint data, ex iis soliditas definiri queat; perinde ac area trianguli ex datis tribus lateribus determinari solet.

PROPOSITIO V. PROBLEMA.

20. *Datis sex lateribus seu aciebus pyramidis triangularis, eius soliditatem inuenire.*

SOLVTIO.

Fig. 5. Sit A B C D pyramidis triangularis, cuius basis triangulum A B C, et vertex D; ac ponantur eius latera: $AB = a$, $AC = b$, $BC = c$, $AD = d$, $BD = e$, $CD = f$. Iam in hedris ADB, et ADC ex D ad bases oppositas demittantur perpendicula D P, et D Q, et in basi A B C ex punctis P, et Q educantur ad latera A B, et A C normales P O, et Q O se mutuo secantes in O, erit recta D O perpendicularis ex vertice D in basin A B C, vnde soliditas pyramidis erit $= \frac{1}{3} DO \times$ aream A B C; at ducta A O, erit $DO = \sqrt{(AD^2 - AO^2)} = \sqrt{(AD^2 - AP^2 - PO^2)}$. Iam ex elementis Geometriae constat, esse, $AP = \frac{aa + dd - ee}{2a}$. et $AQ = \frac{bb + dd - ff}{2b}$ Hinc producta Q O in S, si angulus B A C vocetur $= \alpha$, erit $QS = AQ \text{ tang. } \alpha$, et $AS = \frac{AQ}{\text{cos. } \alpha}$, hinc $PS = \frac{AQ}{\text{cos. } \alpha} - AP$. At cum sit $QS : AQ : AS = PS : PO : OS$, erit $PO = \frac{AQ \cdot PS}{QS} = \frac{PS}{\text{tang. } \alpha} = \frac{AQ}{\text{sin. } \alpha} - \frac{AP}{\text{tang. } \alpha}$, seu $PO = \frac{AQ - AP \text{ cos. } \alpha}{\text{sin. } \alpha}$; tum vero $OS = \frac{AS \cdot PS}{QS} = \frac{PS}{\text{sin. } \alpha} = \frac{AQ}{\text{sin. } \alpha \text{ cos. } \alpha} - \frac{AP}{\text{sin. } \alpha}$, ideoque $QO = QS - OS = AQ \text{ tang. } \alpha$

— A Q

$- \frac{AQ}{\sin.\alpha \cos.\alpha} + \frac{AP}{\sin.\alpha} = \frac{AP - AQ \cos.\alpha}{\sin.\alpha}$. Hinc erit $AO^2 = AP^2 + PO^2 = \frac{AP^2 + AQ^2 - 2 AP . AQ \cos.\alpha}{\sin.\alpha^2}$; ideoque $DO^2 = \frac{AD^2 \sin.\alpha^2 - AP^2 - AQ^2 + 2 AP . AQ \cos.\alpha}{\sin.\alpha^2}$. Verum area trianguli A B C eſt $= \frac{1}{2} ab. \sin.\alpha$, ex quo erit ſoliditas pyramidis $= \frac{1}{6} ab \sqrt{(AD^2 \sin.\alpha^2 - AP^2 - AQ^2 + 2 AP.AQ.\cos.\alpha)} = \frac{1}{6} \sqrt{[aabbdd \sin \alpha^2 - \frac{1}{4} bb (aa + dd - ee)^2 - \frac{1}{4} aa (bb + dd - ff)^2 + \frac{1}{2} ab (aa + dd - ee)(bb + dd - ff) \cos.\alpha]}$. Deinde ex triangulo A B C eſt $\cos.\alpha = \frac{aa + bb - cc}{2ab}$, ideoque $\sin.\alpha^2 = 1 - \frac{1}{4aabb} (aa + bb - cc)^2$, quibus valoribus ſubſtitutis prodibit ſoliditas pyramidis:

$$\frac{1}{12} \sqrt{\left(\begin{array}{l} 4aabbdd - dd(aa + bb - cc)^2 - bb(aa + dd - ee)^2 - aa(bb + dd - ff)^2 \\ + (aa + bb - cc)(aa + dd - ee)(bb + dd - ff) \end{array} \right)}$$

quae terminis euolutis in ſequentem abit formam:

$$\frac{1}{12} \sqrt{\left(\begin{array}{l} aaccdd + aabbee + aabbff + aaddff + bbccdd + bbddee \\ aaccff + aaeeff + bbccee + bbeeff + ccddee + ccddff \\ \quad - aabbcc - aaddee - bbddff - cceeff \end{array} \right)} \\ - a^4 ff - aaf^4 - b^4 ee - bbe^4 - c^4 dd - ccd^4$$

quae adhuc commodius ita exhiberi poſſe videtur:

$$\frac{1}{12} \sqrt{\left\{ \begin{array}{l} + aaff(bb + cc + dd + ee) - aaff(aa + ff) - aabbcc \\ + bbee(aa + cc + dd + ff) - bbee(bb + ee) - aaddee \\ + ccdd(aa + bb + ee + ff) - ccdd(cc + dd) - bbddff \\ \qquad - cceeff \end{array} \right)}$$

Sicque ex datis ſex lateribus a, b, c, d, e, f pyramidis triangularis eius ſoliditas definitur. Q. E. I.

SCHOLION I.

21. Quo ratio, qua in hac expreſſione latera a, b, c, d, e, f inter ſe combinantur, clarius perſpiciatur, notandum eſt, ex iis quatuor formari triangula, ſcilicet

Δ A B C conſtat lateribus a, b, c
Δ A B D - - - - a, d, e
Δ A C D - - - - b, d, f

Δ B C D

Δ B C D - - - - c, e, f

vnde patet, latus a cum ſingulis reliquorum ad triangula conſtituenda concurrere, praeter quam cum latere f, quamob rem haec latera a et f disiuncta appellabo, quia inter ſe non iunguntur; ſimili modo latera b et e erunt disiuncta, itemque latera c et d.

Occurrunt ergo poſt ſignum radicale primo termini ex lateribus disiunctis formati $aaff$, $bbee$, $ccdd$, qui ſunt multiplicati per ſummam quadratorum reliquorum, deinde iidem termini negatiue ſumti multiplicantur per ſummam ſuorum quadratorum, hincque denique ſubtrahuntur producta ex quadratis ternorum laterum cuiusque trianguli.

SCHOLION 2.

22. Formula quoque pro ſoliditate pyramidis inveniri poteſt aliquanto ſimplicior, ſi tria tantum latera in vno angulo ſolido coeuntia dantur, vna cum angulis planis, quos ibi conſtituunt.

Sint enim tria latera in angulo ſolido A coeuntia

$$AB = a, \quad AC = b, \quad AD = d$$

deinde anguli plani:

$$BAC = p; \quad BAD = q; \quad CAD = r.$$

Atque ex his ſoliditas pyramidis erit

$$\tfrac{1}{6}\, abd \sqrt{(1 - \text{cof.}p^2 - \text{cof.}q^2 - \text{cof.}r^2 + 2\,\text{cof.}p.\,\text{cof.}q.\,\text{cof.}r)}$$

quae reducitur ad formam ſequentem:

$$\tfrac{1}{3}\, a\, b\, d \sqrt{\text{ſin.}\tfrac{p+q+r}{2}\,\text{ſin.}\tfrac{p+q-r}{2}\,\text{ſin.}\tfrac{p+r-q}{2}\,\text{ſin.}\tfrac{q+r-p}{2}};$$

vnde patet, vt area prodeat realis, trium angulorum planorum p, q, et r, in angulo quouis ſolido coeuntium binos ſimul ſumtos tertio maiores eſſe debere.

DE MO-