A square acutely triangulated

Article · January 1980

2 authors, including:

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The benefits provided by the greedy algorithm for Heisenberg Boxes of higher order have not yet been quantified. I cannot say whether the greedy algorithm is minimal, but I do believe that the safe algorithm establishes the upper bound for non-redundant drawer search methods. It will be interesting to determine whether the frequency distributions obtained via the greedy algorithm share any of the general trends and singular features associated with the safe algorithm.

For now, uncertainty prevails.

"I am in real trouble, Fred," said his classmate, Tom. "We have a quiz in less than an hour in our Calculation course, and I just dropped my calculator in the mud." Frantically punching at the keys, he went on. "Most of the keys are jammed. I can't get any of the digit keys above 4 to work. Square root works . . . let's see . . . ouch, the Pi key doesn't work, and we are going to need that value for sure. We have to compute the areas of some circles on this quiz, I know. The arithmetic keys are OK. Let's see, what can I do for Pi?"

"I see. You can't key in anything beyond 3.141 . . . the usual approximation, 22/7 won't do . . . what's that other one? Oh, yes, 355/113. That won't do, since your '5' key is stuck. Oh, of course, how about 4xarctan(1)?"

Tom tried it. "No, the Inverse key won't work. Looks like the trig functions are fouled, too. Oh, boy, this is getting desperate."

"Well, if you can manage to get the radius of the circle in by adding numbers with small digits, I guess you could get by if you had a good way to put Pi in. Let me try something on my calculator."

In a few minutes Fred had come up with a way to calculate the value of Pi on Tom's calculator to surprising accuracy with just ten keystrokes. What were the keystrokes he used?

The answer to Tom's plight is the sequence of operations

2143/22 - Sqrt Sqrt
(or 2143 Enter 22 / Sqrt Sqrt on an RPN calculator).

This formula produces the value 3.141592653-, with an error of about 1.0x10**(-9). There is also a similar formula to approximate e, namely

e ≅ 25/43x(Sqrt(10)-1)**2

which uses only the digits 0-5 and produces an error of about 3x10**(-8).

– *Lynn D. Yarbrough*

J. RECREATIONAL MATHEMATICS, Vol. 13(4), 1980-81

A SQUARE ACUTELY TRIANGULATED

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Martin Gardner, in his February 1960 column of Mathematical Games [1], asked for a dissection of the square into eight acute-angled triangles. Furthermore, he wondered whether the dissection could be accomplished with fewer triangles. The following month [2], a solution was given which is essentially the same as that of Figure 1. The same problem recurs in Gardner [3], Coexeter [4], Honsberger [5], and in the Monthly [6].

That the dissection of Figure 1 is unique (up to small pertubations of the two interior points) and implicitly that eight is the smallest number of triangles possible is proven by Lindgren [7]. It should be noted that this dissection into eight triangles is a proper triangulation in that no interior vertex lies on the side of another triangle.

The question arises as to whether dissections of the square are possible into n acute angled triangles where n is greater than eight. Martin Gardner reported having a nine-triangle dissection [1], and in Hoggatt and Jamison [8], they show how to generate dissections for $n \geqslant 10$. However, in both cases all dissections are not proper triangulations. Gardner's nine-triangle dissection apparently has not been published before, and we show it in Figure 2.

In this text we will thus examine the problem of triangulating the square into acuted-angled triangles. By a proper triangulation we mean a subdivision of the square and its interior into non-overlapping triangles in such a way that any two distinct triangles either be disjoint, have a single vertex in common, or have one entire edge in common. And by an interior vertex we mean a vertex (of a triangle) which lies inside the square but not on its boundary. Henceforth,

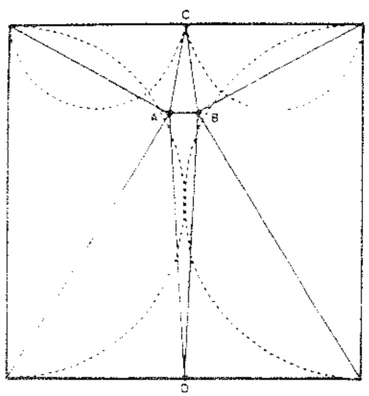

Figure 1.

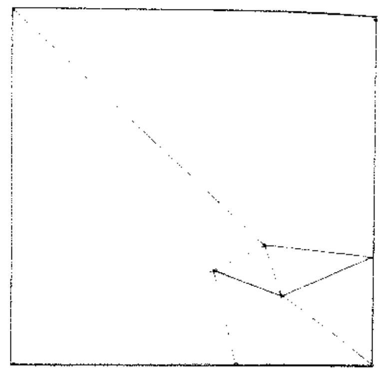

Figure 2.

unless otherwise noted, we will use the term triangulation to mean *proper* triangulation.

We begin with a proof, alternative to that of [7], of the minimality-uniqueness of eight. Then we show there is no triangulation into nine triangles! And finally we demonstrate that there is a triangulation of the square into n acute-angled triangles for all n greater than or equal to ten.

REMARK: If an acute-angled triangle T is contained in a right-angled triangle R then the hypotenuse of R cannot be a side of T.

PROPOSITION: In any triangulation of the square into acute-angled triangles there exist at least two interior vertices. Hence there is no triangulation of the square with fewer than eight acute-angled triangles.

PROOF: We suppose there exists a triangulation of the square, $KLMN$, with no interior vertex. Now the right angle at K must be cut by at least one edge, KK', of the triangulation, and furthermore, K', since it cannot be an interior vertex, must lie on the boundary of the square. However, due to the remark above, this is a contradiction – KK' cannot be the side of the triangle in the triangulation.

Next we consider the case of a triangulation with exactly one interior vertex, P. In order to respect the fact that each triangle is acute, P must be the vertex of at least five triangles and hence the end point of at least five different sides. But so that a second interior vertex not be created, these edges must all terminate on the boundary of the square. There being only four corners of the square one of these five edges must end, at P', somewhere on the square's boundary between two corners, say K and L. Now at least one of the angles $PP'K$ and $PP'L$, let us say $PP'L$, is greater than or equal to $90°$ and so must be cut by a line $P'P''$ such that angle $LP'P''$ is less than $90°$. Since P'' cannot equal P nor can it be, by hypothesis, another interior vertex, then either P'' is on LM or on MN. These two possibilities are excluded by the remark; the former immediately and the latter (P'' on MN) after a repetition of the above argument until a right-angled triangle is generated. (Remember that there are only a finite number of triangles in the triangulation.)

Thus, there are at least two interior vertices. Moreover, as each is the vertex of at least five triangles, there are at least eight triangles in any one triangulation. (The most number of triangles two vertices can share is two.)

A triangulation of the square into eight acute-angled triangles is illustrated in Figure 1. The points C and D are the midpoints of their respective sides; A and B, symmetric with respect to CD, lie outside the four semicircles as shown.

PROPOSITION: There exists no proper triangulation of the square into exactly nine acute-angled triangles.

PROOF: We discuss immediately the possibility of there being three or more interior vertices. Each such vertex must be that of at least five different triangles. Any two vertices can share at most two triangles (by being the endpoints of the same side) and three vertices can determine at most one triangle. Thus even with only three vertices there are at least $15 - 6 + 1$, that is 10, triangles. Hence there exist exactly two interior vertices.

Should there be exactly five edges emanating from each of these two vertices and furthermore should these ten edges be distinct then there would be ten triangles. On the other hand should there be an edge linking the two interior vertices, each still being the endpoint of exactly five edges, then there would only be eight triangles.

A similar analysis eliminates all other possible combinations of edges except for five edges from one vertex and six from the second vertex.

In this latter case eleven triangles would be generated should all of the eleven edges be distinct. Thus, should there exist a triangulation into nine acute-angled triangles, there would be an edge in common between the two interior vertices, thereby creating two triangles sharing this side. And, since there are no other interior vertices, the other vertices of these two triangles must lie on the boundary of the square. Hence these two triangles separate the square and its boundary into two disjoint regions.

Of course the five remaining sides (which emanate from the two interior vertices) must also terminate on the boundary of the square. However as there are only four corners and as the two regions are disjoint, one of these five edges creates a new vertex on the boundary and at that vertex an angle greater than or equal to $90°$. Now by exactly the same argument as in the first proposition, this would imply the existence of a third interior vertex which is thus impossible.

We have shown that no triangulation exists for fewer than eight triangles or for exactly nine triangles. We now establish that a triangulation into n triangles exists where $n = 8$ or $n \geqslant 10$. In the description that follows we give the

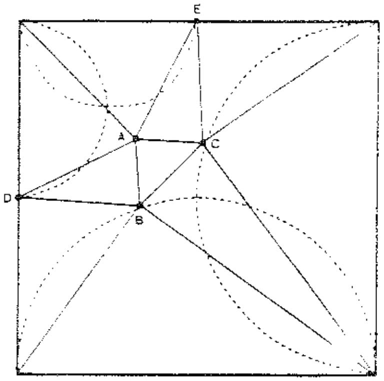

Figure 3.

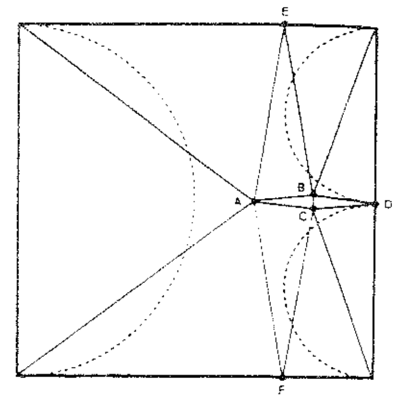

Figure 4.

coordinates of the vertices in a triangulation of the unit square whose bottom left-hand corner is the origin.

FIGURE 1 (8 Triangles)

$A = (\frac{1}{2} - a, \frac{3}{4})$, $B = (\frac{1}{2} + a, \frac{3}{4})$, $C = (\frac{1}{2}, 1)$ and $D = (\frac{1}{2}, 0)$, where $a > 0$ is chosen such that A and B lie outside all of the semicircles.

We can choose the vertices of a triangulation into *$4n + 8$ triangles* ($n \geqslant 1$), as follows:

1. we add n points $A_1, \ldots, A_n$ to the line between A and B:
$$A_i = (\frac{1}{2} - a + \frac{2ia}{n+1}, \frac{3}{4})$$
2. we replace C by $C_1, \ldots, C_{n-1}$ where
$$C_i = (\frac{1}{2} - a + \frac{2ia}{n+1} - \frac{1}{2(n+1)}, 1)$$
3. and D by $D_1, \ldots, D_{n+1}$ where
$$D_i = (\frac{1}{2} - a + \frac{2ia}{n+1} - \frac{1}{2(n+1)}, 0)$$

FIGURE 3 (10 Triangles)

$A = (\frac{1}{3}, \frac{2}{3})$, $B = (\frac{1}{3} + a, \frac{1}{2} - 3a)$, $C = (\frac{1}{2} + 3a, \frac{3}{4} - a)$, $D = (0, \frac{1}{2})$, and $E = (\frac{1}{2}, 1)$. Here the value of a is small enough that B and C lie outside the larger semicircles.

FIGURE 4 (11 Triangles)

$A = (\frac{2}{3}, \frac{1}{2})$, $B = (\frac{5}{6}, \frac{1}{2} + a)$, $C = (\frac{5}{6}, \frac{1}{2} - a)$, $D = (1, \frac{1}{2})$, $E = (\frac{3}{4}, 1)$, and $F = (\frac{3}{4}, 0)$ where a is small enough.

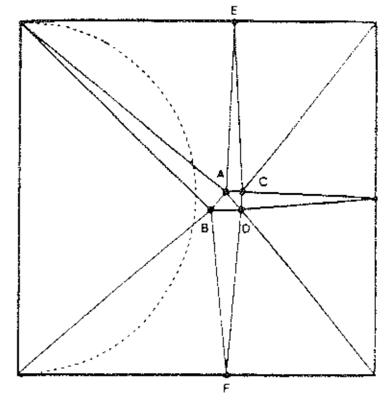

Figure 5.

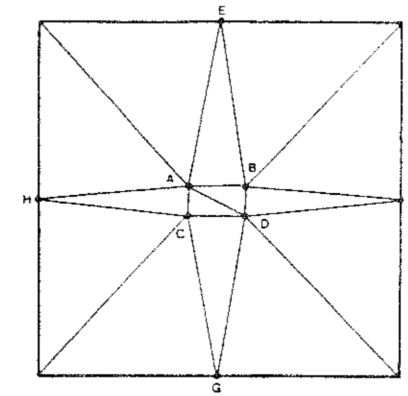

Figure 6.

The vertices in a triangulation into *$4n + 11$ triangles* ($n \geqslant 1$) are A, B, C, D, E, F and $A_i = (\frac{2}{3} - \frac{i}{12n}, \frac{1}{2})$, $E_i = (\frac{2}{3} - \frac{i}{12n} + \frac{1}{24n}, 1)$ and $F_i = (\frac{2}{3} - \frac{i}{12n} + \frac{1}{24n}, 0)$, $i = 1, 2, \ldots, n$.

FIGURE 5 (13 Triangles)

$A = (\frac{1}{2} + \frac{2}{24}, \frac{1}{2} + \frac{1}{48})$, $B = (\frac{1}{2} + \frac{1}{24}, \frac{1}{2} - \frac{1}{48})$, $C = (\frac{1}{2} + \frac{3}{24}, \frac{1}{2} + \frac{1}{48})$, $D = (\frac{1}{2} + \frac{3}{24} - a, \frac{1}{2} - \frac{1}{48})$, $E = (\frac{1}{2} + \frac{2}{24} + \frac{1}{48}, 1)$, $F = (\frac{1}{2} + \frac{1}{12}, 0)$, and $G = (1, \frac{1}{2})$, again a small enough.

FIGURE 6 (14 Triangles)

$A = (\frac{5}{12}, \frac{1}{2} + \frac{1}{24})$, $B = (\frac{7}{12}, \frac{1}{2} + \frac{1}{24})$, $C = (\frac{5}{12} - a, \frac{1}{2} - \frac{1}{24})$, $D = (\frac{7}{12} - a, \frac{1}{2} - \frac{1}{24})$, $E = (\frac{1}{2}, 1)$, $F = (1, \frac{1}{2})$, $G = (\frac{1}{2}, 0)$, and $H = (0, \frac{1}{2})$, a small.

FIGURE 7 (17 Triangles)

$A = (\frac{7}{12}, \frac{1}{2})$, $B = (\frac{9}{12}, \frac{1}{2} + b)$, $C = (\frac{10}{12}, \frac{1}{2} + b)$, $D = (\frac{9}{12} - a, \frac{1}{2} - b)$, and $E = (\frac{10}{12} - a, \frac{1}{2} - b)$. Here a and b are both small with a much smaller than b.

To obtain a triangulation into *$4n + 17$ triangles* ($n \geqslant 1$), we proceed essentially the same as was done with Figure 4.

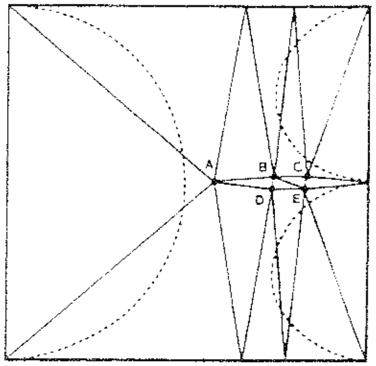

Figure 7.

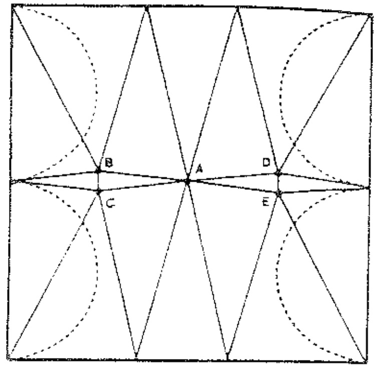

Figure 8.

FIGURE 8 (18 Triangles)

$A = (\frac{1}{2}, \frac{1}{2})$, $B = (\frac{1}{4}, \frac{1}{2} + a)$, $C = (\frac{1}{4}, \frac{1}{2} - a)$, $D = (\frac{3}{4}, \frac{1}{2} + a)$, and $E = (\frac{3}{4}, \frac{1}{2} - a)$,

a small enough.

The same procedure as in Figure 4 and Figure 7 yields triangulations with *$4n + 18$ triangles* $(n \geqslant 1)$.

REFERENCES

1. M. Gardner, *Scientific American*, *202*, p. 150, February 1960.
2. M. Gardner, *Scientific American*, *202*, p. 178, March 1960.
3. M. Gardner, *New Mathematical Diversions*, George Allen & Urwin Ltd., pp. 34-35, 1969.
4. H. M. S. Coxeter, *Introduction to Geometry*, 2nd ed., John Wiley & Sons, p. 152, 1969.
5. R. Honsberger, *Mathematical Gems*, Mathematical Association of America, 1973.
6. Solution to Problem E 1406, *American Mathematical Monthly*, *67*, p. 923, 1960.
7. H. Lindgren, Dividing a Square into Acute-Angled Triangles, *The Australian Mathematics Teacher*, *18*, pp. 14-15, 1962.
8. V. E. Hoggatt, Jr. and F. Jamison, Dissection of a Square into n Acute Isosceles Triangles, *Fibonacci Quarterly Journal*, *6*, pp. 390-392, 1968.

J. RECREATIONAL MATHEMATICS, Vol. 13(4), 1980-81

A FAMILY OF SIXTEENTH ORDER MAGIC SQUARES

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A family of $8^{33}(110)^{17}$ sixteenth order magic squares composed of the first 256 positive integers can be generated from the 880 basic fourth order magic squares listed by Benson and Jacoby [1].

First, select any sixteen of the 880 squares (repetition permitted). The sixteen chosen squares in Figure 1 are ordered according to their upper left elements. The first square of Figure 1 is square (1) in Figure 2. To each element of the second square add 16 to form square (2), to each element of the third square add $2 \cdot 16$ to form square (3), and continue the process until $15 \cdot 16$ added to each element of the sixteenth square forms square (16). Each of these derived squares is magic and remains magic in eight orientations: the square itself, its rotations through 90°, 180°, and 270°, and the mirror images of these four.

To construct sixteenth order magic squares, divide a 16-by-16 grid into sixteen small 4-by-4 grids, thus forming a large 4-by-4 grid of grids. Label the large 4-by-4 grid with the elements of one of the 880 basic fourth order magic squares. In constructing the square in Figure 4, the labelling square used was the pandiagonal square (wherein the elements of the broken diagonals, as well as those of the rows, columns, and unbroken diagonals, sum to the magic constant) of Figure 3.

In each small 4-by-4 grid, place the derived 4-by-4 square, in any of its eight orientations, that has the same identification number as the small grid. That is, proceeding from the upper left corner, in the first square of the large 4-by-4 grid, insert square (1) from Figure 2, in the second square insert square (8) from Figure 2, in the third square insert square (13) from Figure 2, and so on. In the square of Figure 4, the eight derived squares placed in the first two rows were given different arbitrarily chosen orientations, as were those in the last two rows.