Proof. The equation $z^n = 1$ has at most n solutions. On the other hand, the above numbers, n distinct numbers, all satisfy the equation. □

The following property of the exponential function is the basis of Fourier theory:

Theorem A.3. *Let k be an integer. Then*

$$\int_0^1 e^{2\pi i k x}\,dx = \begin{cases} 1 & k = 0; \\ 0 & k \neq 0. \end{cases}$$

Proof. See Exercise A.1.1. □

A.2 The Binomial Theorem

For natural number n and k, with $0 \le k \le n$ we define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The following theorem is fundamental:

Theorem A.4 (The Binomial Theorem). *If n is a natural number, then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. The proof is an easy induction and ultimately relies on the fact that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

□

We now use the Binomial Theorem to prove the following theorem:

Theorem A.5. *For $k, y \in \mathbb{N}$ define*

$$\sigma_k(y) = \sum_{m=1}^{y} m^k.$$

Then there is a polynomial $f_k(x)$ with rational coefficients with leading term $x^{k+1}/(k+1)$ such that

$$\sigma_k(y) = f_k(y).$$

Proof. We will prove the theorem by induction. For $k = 1$ we have

$$\sum_{m=1}^{y} m = \frac{1}{2}y^2 + \frac{1}{2}y.$$

Now suppose we know the theorem for every $l < k$. By the Binomial Theorem

$$(m+1)^{k+1} - m^{k+1} = \sum_{j=0}^{k-1} \binom{k}{j} m^j.$$

As a result

$$(y+1)^{k+1} - 1 = \sum_{m=1}^{y} \{(m+1)^k - m^k\} = \sum_{j=0}^{k} \binom{k+1}{j} \sigma_j(y).$$

Consequently,

$$\binom{k+1}{k} \sigma_k(y) = (y+1)^{k+1} - 1 - \sum_{j=0}^{k-1} \binom{k+1}{j} f_j(y).$$

By the induction hypothesis the right-hand side is a polynomial of degree $k+1$ with leading term y^{k+1}. Once we observe

$$\binom{k+1}{k} = k+1$$

the theorem follows. □

Corollary A.6. *For all natural numbers k,*

$$\sigma_k(y) = \frac{y^{k+1}}{k+1} + O(y^k).$$

A.3 The Pigeon-Hole Principle

The Pigeon-Hole Principle is the following intuitively obvious statement: If we distribute n balls among m boxes, with $n > m > 0$, then at least one box will end up with more than one ball. Stated differently, if we have n pigeon trying to get in m pigeon-holes, with $n > m > 0$, then at least one of the pigeon-holes will have two pigeons in it, hence the title *The Pigeon-Hole Principle*. The Pigeon-Hole Principle is also known as Dirichlet's Box Principle. Dirichlet (1834) used this principle to prove a theorem about rational approximation to irrational numbers. We present this theorem in Example A.11 below. The Pigeon-Hole Principle is an extremely useful statement with many applications. In this appendix we give a proof of this statement using mathematical induction. We then give several applications. The appendix ends with a few standard problems.

The Pigeon-Hole Principle should be thought of as a statement about functions. Let A be the set of pigeons and B the set of pigeon-holes. Then the process of sending pigeons to pigeon-holes is a function from $A \to B$. The technical statement of the Pigeon-Hole Principle is the following:

Theorem A.7. *Let A, B be finite sets with $\#A > \#B$. Then there are no injective maps $f : A \to B$.*

Proof. We will prove this by induction on $\#B$. If $\#B = 1$, and $\#A > 1$, it is clear that we cannot have an injective function $f : A \to B$ as there is only one option for the image of the function f. Now suppose $\#B = k \geq 2$ and that we know the theorem for every set of size $k-1$. Suppose A is a set with $\#A > \#B$ and let $f : A \to B$ be an injective map. Pick an element $b \in B$. Since f is injective, $f^{-1}(b)$ consists of a single element $a \in A$. Then $\#(B - \{b\}) = k-1$, and the restriction of f to $A - \{a\}$ gives a function $\tilde{f} : A - \{a\} \to B - \{b\}$. By the induction hypothesis this function $\tilde{f}$ is not injective, hence the original function f could not be injective. □

Similarly one can show that if we have sets A, B with $\#A > k\#B$ for some natural number k, then there is at least one element $b \in B$ such that

$$\# f^{-1}(b) \geq k+1.$$

We now give some examples.

Example A.8. Of every eight people, there are at least two who are born on the same day of the week. Of every fifteen people, there are at least three born on the same day of the week.

Example A.9. Of every $n+1$ integers, there are at least two with difference divisible by n. In order to see this write $\mathbb{Z}$ as the disjoint union of the following n subsets $\mathbb{Z}_a$, $0 \leq a \leq n-1$. For each a, let $\mathbb{Z}_a$ be the set of integers k such that $k \equiv a \bmod n$. Since we have $n+1$ elements and n sets $\mathbb{Z}_a$, there is an a with the property that $\mathbb{Z}_a$ contains at least two elements x, y of the set. Since $x \equiv a$ and $y \equiv a$, it follows $x \equiv y \bmod n$ and consequently, $n \mid x - y$.

Example A.10. We will show that of every five distinct real numbers at least two of them satisfy

$$0 < \frac{a-b}{1+ab} < 1.$$

Let the five numbers be $a_1, \ldots, a_5$. Since the map $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection, there will be five angles $\theta_i \in (-\pi/2, \pi/2)$, $1 \leq i \leq 5$, such that $a_i = \tan\theta_i$. Now divide up the interval $(-\pi/2, \pi/2)$ to four subintervals $(-\pi/2, -\pi/4]$, $(-\pi/4, 0]$, $(0, \pi/4]$, and $(\pi/4, \pi/2)$. Since we have five θ_i's and four subintervals, by the Pigeon-Hole Principle at least two of them will be in the same subinterval. This means that there are indices i, j such that

$$0 < \theta_i - \theta_j < \pi/4.$$

Since tan is monotone increasing on the interval $(-\pi/2, \pi/2)$, we have

$$\tan 0 < \tan(\theta_i - \theta_j) < \tan(\pi/4).$$

Now we recall $\tan 0 = 0$, $\tan(\pi/4) = 1$, and that for angles α, β,

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}.$$

We finally get

$$0 < \frac{a_i - a_j}{1 + a_i a_j} < 1$$

and we are done.

Example A.11 (Dirichlet). If α is an irrational number, then there are infinitely many rational numbers p/q, with $\gcd(p, q) = 1$, such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Let n be a natural number. We will prove that there is a rational number p/q such that $1 \leq q \leq n$ with the property that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qn}. \tag{A.1}$$

It is not hard to see that the main claim of this example follows from this statement. Equation A.1 is equivalent to the existence of a pair of integers (p, q) with $1 \leq q \leq n$ such that

$$|q\alpha - p| < \frac{1}{n}.$$

Consider the fractional parts $\{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}$. These are n numbers in the interval $(0, 1)$, and never a rational number, as otherwise α would be a rational number. In particular, each of them lands in the one of the following *pigeon-holes*: $(0, 1/n)$, $(1/n, 2/n)$, $\ldots$, $(1 - 1/n, 1)$. If one of the $\{k\alpha\}$ falls in the first of these intervals $(0, 1/n)$, then we have $0 < \{k\alpha\} < 1/n$, which gives $0 < k\alpha - [k\alpha] < 1/n$. This verifies the assertion with $p = [k\alpha]$ and $q = k$. If none of the fractional parts falls in the first interval, then we have n fractional parts in $n-1$ intervals. By the Pigeon-Hole Principle two of the fractional parts, $\{k\alpha\}$ and $\{l\alpha\}$ say, will be in the same interval. Without loss of generality assume $k > l$. Since the length of each of the intervals is $1/n$ we will have

$$|\{k\alpha\} - \{l\alpha\}| < 1/n.$$

The left-hand side of the inequality is equal to

$$|k\alpha - [k\alpha] - l\alpha + [l\alpha]| = |(k-l)\alpha - ([k\alpha] - [l\alpha])|.$$

The result follows with $q = (k-l) < n$ and $p = [k\alpha] - [l\alpha]$.

Exercises

A.1.1 Use Theorem A.1 or any other method to prove Theorem A.3.

A.1.2 Use Theorem A.1 to give a proof for the addition formula for sine and cosine:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

A.1.3 Compute $\cos\frac{\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{3\pi}{7}$.

A.1.4 Compute the value of $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7}$.

A.1.5 Let $\eta_1 = 1, \eta_2, \eta_3$ be the three third roots of 1 in $\mathbb{C}$. Find a formula for the value of $\eta_1^n + \eta_2^n + \eta_3^n$ for $n \in \mathbb{Z}$.

A.2.1 Show that for $n \in \mathbb{N}$,

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n, \quad \sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$$

A.2.2 Prove that for all natural numbers n,

$$\sum_{k=0}^{n}\binom{k}{r} = \binom{n+1}{r+1}.$$

A.2.3 Show that for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}.$$

A.2.4 Prove that for all natural n

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k} = \frac{-1}{n}\binom{2n}{n}.$$

A.2.5 Prove the identity

$$\sum_{k=1}^{n}k2^{-2k}\binom{2k}{k} = \frac{n(n+1)}{3\cdot 2^{2n+1}}\binom{2n+2}{n+1}.$$

A.2.6 Show that for all $n \in \mathbb{N}$, $n^2 \mid (n+1)^n - 1$.

A.2.7 Show that for all natural numbers n, k,

$$\frac{1}{k+1}n^{k+1} < \sum_{r=0}^{n}r^k < \left(1+\frac{1}{n}\right)^{k+1}\frac{1}{k+1}n^{k+1}.$$

A.3.1 Show that if we have six numbers from the set $\{1, 2, \ldots, 10\}$ two of them add up to an odd number.

A.3.2 Show that if we have a subset $A \subset \{1, 2, \ldots, 100\}$ with ten elements, then the set A has disjoint subsets S, T whose elements have the same sum.

A.3.3 Show that if we choose a subset $S \subset \{1, 2, \ldots, 2n\}$ with $n+1$ elements, then there are at least two integers $x, y \in S$ such that $x \mid y$.

A.3.4 Show that if we choose five points in a unit square, there are at least two of them that are at most $\sqrt{2}/2$ apart.

A.3.5 Show that of every group of n people there are two with an identical number of friends in the group.

A.3.6 Suppose we have an infinite array of natural numbers $(a_{ij})_{i,j\in\mathbb{N}}$ with the property that $a_{ij} \le ij$. Show that for every natural number k, there is at least one natural number m which is repeated at least k times in the array.