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(on the occasion of his centenary)

On 28 June, 1975 a hundred years have passed since the birth of the outstanding French mathematician Henri Lebesgue. His main work belongs to the first quarter of this century, but his influence on the development of mathematics is evident to the present day. The ideas he introduced and the methods he worked out were to become indispensable scientific tools that have been used ever since in the most diverse areas of modern pure and applied mathematics.

Henri Léon Lebesgue was born in the town of Beauvais; his father was a typographer and died only a few years later. Even when Lebesgue was a young boy in primary school, his extraordinary abilities were noticed. Later the initiative of one of his teachers, combined with the backing of the mayor of Beauvais, obtained for the talented young pupil a grant of municipal funds to pay for his continued studies, first at the town college and then at one of the Lycées of Paris. After leaving the Lycée, Lebesgue was admitted as a student to the École Normale Supérieure in Paris. He received the title of secondary school teacher in 1897 and started his teaching career at the Lycée of Nancy. There he wrote a number of scientific papers, in particular, he worked on his dissertation. A few years later he was invited to become Maître de conférences at the College of Science of the University of Rennes (1902–1906), and later he became a lecturer at the university of Poitiers (1906–1910).

The first decade of this century became for Lebesgue a period of intensive scientific productivity. In these years he published more than forty papers, including the books [8] and [14], in which he first presented practically all his main scientific achievements, and as a result he was recognized as one of the outstanding mathematicians of his time. Lebesgue's later publications, although still rather extensive, could not add very much to the scientific reputation that he had already acquired through his research activities during this first decade of the century.

The wide recognition of Lebesgue's scientific accomplishments brought with it an invitation to Paris. Lebesgue started to teach at the Sorbonne in 1910, and he became a Professor there nine years later. In 1920 he occupied the Chair of Mathematics at the Collège de France, where he continued to teach until the end of his life. In 1922 Lebesgue was elected a Member of the Paris Academy of Science, in 1924 he became an Honorary Member of the Royal Society of London, and subsequently of many other scientific institutions in France and abroad. Generally, academic honors were bestowed on him only when he had already passed the zenith of his power of creative inspiration.

To Lebesgue can be applied in full measure the words of his younger contemporary and colleague in the profession, the Italian scholar D. F. Tricomi: "His scientific works are his biography". Mathematics, including the teaching of it and its history almost exhausted the sphere of his intellectual interests. To be sure, his outwardly quiet life as a scholar and a seemingly happy family man (he was married and had two children) was often disturbed by dramatic events. It is probable, for example, that the more than cool attitude towards his thesis [3] of some great French scholars such as Darboux caused him considerable distress. His hostile relations with E. Borel also undoubtedly poisoned many moments in the lives of them both; their quarrels were collisions between two different human temperaments, in a sense two different attitudes towards life or even between two dissimilar moral codes. Also, it can hardly have been with a light heart that Lebesgue, towards the end of his life, experienced how his native country was enslaved by the German invaders. There were, of course, both bad and good incidents on the personal plane. But all this proved to be of no lasting importance. What did last were the scientific values created by Lebesgue. They were lasting not in the sense that they exist in a form created once and for all by inspiration from above. Instead, they must be regarded only as links in the chain of progressive evolution of mathematics, but links that cannot be severed and without which the whole mathematical science would certainly not look the way it does today,

Lebesgue worked nearly until the moment of his death, on 21 July 1941. He made significant contributions to the most diverse fields – to the theory of functions and sets, to the theory of surfaces and variational calculus, to topology and elementary geometry, to the theory of integral equations and functional analysis, to the theory of Dirichlet's problem as well as to mechanics, but also to the history of mathematics and to the methods of mathematical education. But most of all, he has become famous for his work on the theory of real functions and sets. In what follows we give a brief account of his main results in this direction.

One of the important innovations due to Lebesgue was his introduction of the concept of measurable sets and functions. True, Borel and Baire in 1898 had put forth the idea of B-measurable sets and functions, and the former even considered more general sets (see [26], 48). This fact made Denjoy ([27], 5–6) inclined to attribute to Borel priority with respect to the creation of the concept of measurable sets in the sense of Lebesgue. But Borel presented his idea without much emphasis, and certainly not in a general form. Lebesgue, on the other hand, ([2], [3], 237–238, 258) from the same point of departure went on to introduce measurability of sets and functions from a general point of view as a necessary condition for the applicability of his construction of the integral by means of sums.

The various classes of functions that had been utilized in mathematical analysis before the time of Lebesgue, analytic functions, continuous functions, functions of bounded variation, B-measurable functions, etc., each of these classes was, of course, of great interest in one connection or another, but they also proved to be disappointing to analysts, in the following sense: as soon as they had been introduced, or very soon afterwards, people realized that the ordinary operations of analysis might lead beyond the limits of that particular class of functions. The limits of even the most general of the classes listed, that of B-measurable functions, were exposed by Lebesgue as early as in 1905 (see [13], 213–215), but they became even clearer after 1917. That was the year when the A-operation, recently introduced by P. S. Aleksandrov and Hausdorff, was applied by Suslin to the construction of sets (and functions) that are not B-measurable.

However, not even the most general form of convergence, nor any of the summation methods (not to speak of any differentiation operation; in short, none of the analytic operations) would lead to a non-measurable function in the sense of Lebesgue. Thus, in fact the situation is that even today, many decades after Lebesgue's introduction of his concept of a measurable function, work in all branches of analysis (whether it be an abstract theory of differentiation and integration, or an extremely general theory of representation by series, etc.) still proceeds essentially within the framework of measurable sets and functions.

True, examples of non-measurable sets (or functions) in the sense of Lebesgue were constructed as early as 1905 by Vitali, and in 1908 by van Vleck, and since then many more examples of a similar nature have been found; the question of measurability, in the sense of Lebesgue, of the various classes of projective sets introduced in 1928 by Luzin and V. K. Sierpiński turned out to be very difficult to answer. Moreover, there are several papers in which the authors, having worked out one problem or another, try to get rid in their arguments of the restriction of measurability of the functions or sets under discussion. But all the research of this kind remains, for the time being, outside the main stream of analysis, and it is far from obvious that it will lead anywhere. In any case, no one has yet identified really tangible classes of functions or sets, more general than that of measurable functions and sets, but amenable to the traditional apparatus of analysis.

Measurable sets and functions were, in fact, introduced by Lebesgue in 1901 in the note [2], where he established at the same time that the most general limit process then known – convergence almost everywhere – does not lead outside the class of measurable functions. However, in the same note he did not yet make a distinction between measurable and summable functions. He did not develop explicitly the difference between the two concepts even in his dissertation, although he provided there an example of a measurable but not summable function ([3], 217). Only in 1903 in [6] did he make conscious distinction between summability and measurability of a function; there he formulated the definition of measurability of a function in the form in which it was to be used afterwards by him and others; there he also observed that every measurable function has the *C*-property and, conversely, that every function having the *C*-property is also measurable; in other words, in [6] Lebesgue formulated the theorem that was later proved by Luzin in 1912.

All subsequent contributions to analysis by Lebesgue revolved within the framework of measurable sets and functions, as he did not make use of the axiom of choice. Of course, the concept of measurability in the sense of Lebesgue was later generalized (by Radon, Fréchet, and others), but the hard core that was due to Lebesgue always remained the same.

Lebesgue became famous first of all for his theory of integration; he developed it to such completeness and perfection and found such diverse areas of application for it that from then on it became an anachronism to study or use any of the preceding theories.

Lebesgue's first papers on the theory of integration [2], [3], [8], [15–17], and [19] were devoted to the problem of integration of functions of one real variable. The only exception is that in his dissertation [3] he obtained a number of results for the many-dimensional case, of which we may mention a proof of Fubini's theorem for bounded functions ([3], 276–281). On the other hand, the basic purpose of his long paper [20] "Sur l'intégration des fonctions discontinues" was to extend the results of his one-dimensional theory to the case of several variables. To do this he used concepts that led far beyond the limits of straightforward extensions of his previous results. As a matter of fact, he began in this way the construction of a new mathematical theory, important in its own right: the theory of set functions.

Lebesgue was not the first to consider set functions. As early as in 1887, an attempt was made by Peano, inspired by even earlier suggestions of Cauchy, to develop a theory of set functions in an even more general situation than that considered by Lebesgue. But Peano's efforts, in spite of their great historical interest, never received much attention from other mathematicians. Apparently Lebesgue was not aware of them nor of Cauchy's contributions to the subject. He presented the idea of set functions in an absolutely original and very clear fashion, and went on to study different possible forms of such functions, as well as the connection between set functions and his theory of integration. After publishing his paper [20] on the theory of set functions, he began an intensive effort to work out a great number of applications. The theory has now grown into a vast mathematical discipline, leaving its own distinct imprint on the theory of functions as well as on the theory of probability, functional analysis and geometry, indeed on our whole way of mathematical thinking. But the main source of the theory was definitely Lebesgue's own paper "Sur l'intégration des fonctions discontinues". Let us mention here only one more result contained in that paper:

Lebesgue came to consider the indefinite integral as a set function ([20], 381), and he introduced the notion of the derived numbers of a set function with respect to a regular family of sets ([20], 395). He then proved ([20], 399) the following theorem: an absolutely continuous additive set function has a derivative that is defined and finite almost everywhere, and it is an indefinite integral of any function that is equal to this derivative at all points where the latter is defined and finite, but takes arbitrary values elsewhere. This theorem was, in a sense, the climax of Lebesgue's theory of multiple integration, since it carries over the Newton-Leibniz theorem to the many-dimensional case and made it possible to establish that the operations of differentiation and integration are connected similar to the situation in classical analysis. It also laid the foundation for the celebrated Radon-Nikodym theorem,

"On several occasions attempts were made to generalize the old process of integration of Cauchy-Riemann, but it was Lebesgue who first made real progress in the matter. At the same time, Lebesgue's merit is not only to have created a new and more general notion of integral, nor even to have established its intimate connection with the theory of measure: the value of his work consists primarily in his theory of derivation which is parallel to that of integration. This enabled his discovery to find many applications in the most widely different branches of analysis . . ." ([29], V).

Among the many achievements of Lebesgue in the theory of differentiation, it should first of all be mentioned here that he invented the concept of differentiability almost everywhere, studied it at great depth, and found a wealth of applications. Although he did not actually give an independent definition of the concept, he used it constantly ever since the publication in 1901 of his note [2]. He employed it, in particular, in his proof (see [5], and also [8], 128) of one of the fundamental theorems of the theory of real functions, about the differentiability almost everywhere of every continuous function of bounded variation.[1])

1) The theorem was extended in 1911 by G.C. and W.H. Young to the case of arbitrary functions of bounded variation, with a proof that did not make use, as Lebesgue's proof had done, of the concepts of integral and transfinite numbers.

Another fundamental contribution by Lebesgue to the theory of differentiation, as we have already mentioned, is the introduction of the notion of derivative of a set function with respect to a measure, and his demonstration of how this new concept is connected with the concept of integral. In 1910 he extended the one-dimensional theorem about differentiability almost everywhere of a function of bounded variation to the case of additive set functions of bounded variation in Euclidean spaces of any finite number of dimensions (see [20], 408–425).

Lebesgue did not pioneer the study of differentiation of one point function with respect to another. But to him is due the formulation of some problems in this field, as well as many other interesting results (see [25], 234–251). Let us point out, for example, that he was the first to formulate the definition of an approximate derivative of one function with respect to another[1] as (ibid. 240) and on the basis of it he began to develop the theory of the Denjoy-Stieltjes integral; he also raised the problem (ibid., 255) of the uniqueness of a primitive function with respect to an arbitrary continuous function, a problem that was later solved by Petrovskii (1929, 1934) and Cacciopoli (1934).

The theory of trigonometric series underwent at the hands of Lebesgue the same kind of transformation as the theory of integration and differentiation.

The sequence of papers by Lebesgue on this subject starts in 1902 with the note [4], which contains a generalization of the uniqueness theorem for trigonometric expansions in the form due to du Bois-Reymond, to the case when the coefficients of the series are expressed in terms of the Lebesgue integral. Then followed the papers [7] and [9]–[11], which were crowned by his famous "Leçons sur les séries trigonométriques" [14].

Lebesgue's main achievement in this field was the reconstruction of the theory of Fourier series on the basis of his new concept of integral. Just as had happened in the theory of integration, so after Lebesgue's work on the subject, the Fourier series with all its connotations started to be thought of as the Fourier-Lebesgue series. At the same time, Lebesgue's publications on the subject contained a wealth of new results, ideas and methods that became starting points for a great number of later investigations. Let us indicate only a few of them.

Lebesgue made a deep analysis ([11], 251–260) of the various conditions for convergence of Fourier series that had been proposed, at different times, by Dirichlet, Lipschitz, Jordan and Dini, and he could show (ibid., 267–269) that they are all contained as special cases in a new general convergence criterion which he established.

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1) Khinchine and Denjoy had already introduced the notion of an approximative or asymptotic derivative, but they considered only differentiation with respect to an independent variable.

Generalizing Fejér's theorem (1900) on the summability by the Cesàro method of the Fourier series of a bounded R-integrable function $f(x)$ at points of continuity or discontinuity of the first kind, to a theorem on the summability almost everywhere, by the same summability method, of the Fourier series of every L-integrable function ([11], 277), Lebesgue also extended it to the Riemann method (ibid., 280). During the following year he made a more detailed study ([14], 89–96) not only of these methods, but also of Poisson's summability method. Of course, Fejér was the first to consider the problem of summability of a divergent Fourier series, but it was not until after the publication of Lebesgue's book [14] that summability methods were more widely studied and applied in the theory of trigonometric series. The broader question on the analytic representation of functions belongs to the same context.

At the turn of the century, and particularly after Baire's investigations, it came to be assumed that the most general way of representing functions in terms of analytic expressions was to define them as limits (simple or iterated, where the iteration could be continued transfinitely) of sequences of continuous functions (or polynomials), convergence to a limit being understood as convergence everywhere. As a result it would turn out that the most general functions having an analytic representation are the B-functions (Baire-functions). Lebesgue himself for some time believed this to be the case. However, in the course of his investigations on trigonometric series, his point of view underwent a substantial change: he came to consider an analytic expression as a representation of a function also when convergence takes place only almost everywhere; he then extended representability to summability in one sense or another, or even summability almost everywhere. This is quite noticeable in his 1905 paper already mentioned ([11], 272, 277, 279), but it became particularly clear in his "Leçons sur les séries trigonométriques" [14]. This extension of the meaning of the phrase "analytic representation of a function" turned out to be very fruitful in the subsequent developments.

Let us also mention here that in his research on Fourier series Lebesgue was led to introduce[1] the important concept of the density of a set at a point ([11], 266), which subsequently acquired much independent interest.

Lebesgue continued his study of Fourier series also after 1906. In particular, in his 1910 paper "Représentation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz" [21] he considered questions of the rate of convergence of Fourier series of functions satisfying Lipschitz conditions of various kinds.

Lebesgue made very significant contributions to the theory of B-sets (Borel-sets) and B-functions (Baire functions).

The B-sets were introduced by Borel in 1898 ([26], 46–48). There he

1) This concept was mentioned by Lebesgue for the first time in 1903 ([6], 1229, footnote 1).

formulated his definition of a measure and proved the measurability in this sense of some classes of linear point sets, but he did not, in fact, investigate them in detail either in this book or in his following papers. In essence, the theory of Borel sets was not developed to any significant degree until 1905.

The situation with regard to the B-functions was entirely different. They were also first defined in 1898, by Baire, in his note [30], but in contrast to Borel, he began an intensive development of the theory of these functions. The matter was soon taken up by Lebesgue and Borel, and by 1905 there existed a fairly well developed theory of Baire functions.

It is an interesting fact that neither Borel, who defined the B-sets, nor Baire, for all his profound insights into the properties of B-functions, observed the connections between these two mathematical objects. On the contrary, Lebesgue, first in his dissertation ([3], 257–258) and later in his book ([18], 111–112), threw the first bridge between them, by showing that every Baire function $f(x)$ has the property that its corresponding set $E[f(x) > a]$ is Borel-measurable. And in 1905, in his fundamental memoir "Sur les fonctions représentables analytiquement" [13], Lebesgue forged an inseparable link between Borel sets and Baire functions, thereby advancing the theory of Borel sets considerably. Here he also constructed and examined (ibid., 156–165) a classification of Borel sets, parallel to Baire's classification of functions. Later, by changing the definition of Baire functions to the requirement of Borel measurability of the sets $E[f(x) > a]$ and calling functions with this property Borel measurable (ibid., 167), Lebesgue established (ibid., 168–170) that for a function to occur in the Baire classification it is necessary and sufficient that it is Borel measurable.[1]

Lebesgue's memoir just mentioned is extraordinarily rich in original ideas, methods of constructing functions and sets, and ways of reasoning; their significance is described in detail by Luzin ([31], 249–266) and we do not dwell on this, except to remark that here Lebesgue established (ibid., 191) an analogue for functions of an arbitrary class, to Baire's theorem on functions of the first class; he also constructed here the first example of a function to which Baire's classification cannot be applied (ibid., 213–215). This memoir was going to become the starting point for the investigations of Luzin, Sierpiński, Suslin, and many other scholars, on the properties of Borel sets and Baire functions, as well as more general mathematical objects. "In particular, the whole theory of analytic sets has its origin in this memoir" (Luzin [31], 249).

Many other ideas, methods, concepts, and results are connected with the name of Lebesgue. Let us mention briefly a few of them.

To the directions of Lebesgue's research we must add his contributions to the theory of surface area and the theory of singular integrals. In these fields he had had some famous forerunners, but precisely in his papers [3]

1) Baire defined B-functions by means of iterated limit passages everywhere, over sequences of continuous functions.

and [19], these theories were first worked out systematically, and Lebesgue is usually regarded as their founder. How long-lived Lebesgue's ideas were is testified, for example, by the fact that his definition of area of a surface, dating back to 1902, was even in 1943, according to the judgement of T. Radó, which essentially is still valid today, "the most fruitful of the many definitions proposed to date" ([32], 141).

The principle of negligibility of various point sets in considering one problem or another in analysis or the theory of functions came into use at a very early stage. Originally, only finite point sets were regarded as negligible, but later, about at the time of Cantor, also various kinds of infinite sets (reducible, countable, etc.). Towards the end of the nineteenth century, Baire in considering certain problems in the theory of functions was led to neglect sets of the first category. Lebesgue, starting with the note [2] and his dissertation [3], began to make frequent use of the principle of negligibility of sets of measure zero. He gave an explicit formulation of this principle in 1903 ([6], 1229), basing it on an argument due to Baire, and then, as we have already said, applied it systematically to problems of differentiation, integration, convergence, summability, etc., in a word, to the most diverse problems of the metric theory of functions. It was just, through the work of Lebesgue, that the principle of negligibility of sets of measure zero became one of the essential methods of the theory of functions of a real variable. Of course, every time that Lebesgue's concept of measure is generalized in some direction, the principle has to be modified accordingly, but the scheme of its application basically remains the same.

Lebesgue was not the first to use the sets $E[f(x) > a]$, $E[f(x) = a]$, $E[f(x) < a]$; they were used, in particular, by R. Baire. But only at the hands of Lebesgue, again beginning with his note [2] and the dissertation [3], these sets became powerful tools for the investigation of properties of functions. Lebesgue used them in his construction of approximating sums for integrals, in his definition of the general concept of a measurable function, in his studies of B-functions, and in many other contexts. The way he looked at it, "donner une fonction, ... , c'est donner les ensembles $E[a < f(x) < b]$", as he wrote in 1918 ([23], 236). Beginning with this then the paper and up to the present day the use of these sets has proved to be a necessary condition for obtaining very many results in the theory of functions of a real variable.

The categories method of, together with the methods of set and measure theory, are used in proving many existence theorems. It also goes back to Baire, but apparently was first emphasized in Lebesgue's 1917 paper [22] as a general method of proof. However, the category method became more widely used only after the work of Polish mathematicians during the thirties.

We should mention here also Lebesgue's method of chains of intervals (see [25], 273–279), which does not play a big role in the theory of functions, but was used repeatedly by Lebesgue to establish many important results. He applied method, for example, in [8] (63, 79, 121–122, 126), in [15] (5–6), [16] (96–97), and [17] (286–288), for the proof of some fundamental theorems in his theory of integration.

An important theorem in the theory of functions of a real variable states that an almost everywhere convergent sequence of measurable and almost everywhere finite functions is also convergent in measure. The special case of an everywhere convergent sequence had already been proved in 1885 by Arzelà. Lebesgue, who was not aware of this fact, used it in 1902 ([3], 259), stated it explicitly in 1903 ([6], 1229), and proved it in 1906 ([14], 10). As early as in 1903, Lebesgue mentioned explicitly ([6], 1229) as a corollary to this theorem the statement that later became known as Egorov's theorem.

An important role in Lebesgue's theory of integration is played by the theorem about termwise integration of convergent series of functions with uniformly bounded remainders. The theorem was first stated and proved by Arzelà in 1885 and 1900 for Riemann-integrable functions, and independently by Osgood in 1897 for continuous functions. Lebesgue who did not know about the results of Arzelà, took his departure from Osgood's theorem and generalized it to the case of summable functions ([3], 259–260; [8], 114); later in 1908–1910 ([18], 12–13; [19], 49–50; [20], 375–376) he extended this result to the theorem that it is legitimate to integrate termwise a convergent sequence of summable functions, provided that the terms of the sequence do not exceed in absolute value some fixed summable function.

One of the most important theorems of nineteenth century analysis was the theorem asserting that every continuous function has a primitive. It was usually proved by means of the concept of the definite integral or by applying Weierstrass' theorem on the representation of every continuous function by a uniformly convergent series of polynomials. However, at that period the integral $\int_a^b f(x)dx$ was usually defined in terms of the difference $F(b) - F(a)$, where $F'(x) = f(x)$, and this definition was regarded as equivalent to the definition of the integral of a continuous function as the limit of an approximating sequence of Cauchy sums. Arguments of this kind, however, contains a logical circle: to regard the integral $\int_a^b f(x)dx$ as the difference of certain values of a primitive function, one has first to establish the existence of a primitive $F(x)$, but this is proved by means of the concept of integral, defined either as the limit of its approximating sums or by Weierstrass' theorem, which is usually proved by recourse to the concept of an integral. Lebesgue succeeded in avoiding this circle argument, when he showed in [12] in 1905, without reference to the concept of the definite integral, that every continuous function has a primitive, relying on his earlier of the Weierstrass theorem [1], in which this concept does not figure at all. The analogous problem for more general primitive functions was considered by Lebesgue [24] in 1926.

Our list of Lebesgue's results in the theory of real sets and functions is by no means complete. To it we could add theorem on convergence to zero of the Fourier coefficients of summable functions ([7], 471–474), the theorem on the representation of a summable function by a singular integral at its points of continuity ([19], 69–71), a generalization of Vitali's covering theorem ([20], 391–394), the theorem on density points of measurable sets ([20], 406–407), the theorem on the representation of an additive function as a sum of an absolutely continuous and a singular function ([20], 413–415). As examples of less important results we could point to the theorem that functions with bounded derivative numbers are of bounded variation ([8], 73), or the theorem giving necessary and sufficient conditions for a given function to have an indefinite Riemann integral ([19], 40–42).

Some ideas expressed by Lebesgue, but developed by other mathematicians are also not without interest. For instance, already in his dissertation ([3], 272) he tried to generalize his integral in the direction realized in 1912 by Denjoy; but he did not then evolve his approach in detail, a fact that he later came to regret ([23], 205). In 1904, in his "Leçons sur l'intégration et la recherche des fonctions primitives" ([8], 129), he actually introduced the concept of an absolutely continuous function, but he did not recognize its significance properly, which was done by Vitali (1905). In 1909 he outlined the idea ([19], 30) of an approach to his integral by Riemann sums, using some restrictions on the nature of the partitioning of the integration interval and the choice of points at which the function values are computed; this definition of the Lebesgue integral and even more general integrals, by means of Cauchy-Riemann sums was later investigated by Denjoy (1919, 1931); Lebesgue himself used a similar modification of the partitioning of the integration interval to obtain the Riemann-Stieltjes integral ([25], 253–256). In the same 1909 paper one finds the idea of weak convergence of sequences of measurable functions ([19], 55), an idea that was developed in the next year by F. Riesz.

Of considerable interest are also Lebesgue's thoughts about transfinite numbers, the concept of a function, Zermelo's axiom, but it would be difficult to give a brief account of them. Therefore, let it be sufficient to say that they too left their definitive imprint on the development of the theory of sets and functions during the twentieth century.

All in all, Lebesgue's contribution to the construction of the theory of functions of a real variable, and indirectly to almost all areas of present-day mathematics, is of exceptional magnitude. In the history of mathematics it has not happened often that the efforts of a single scholar have contributed so much to the creation of a large mathematical discipline. And if one adds to this fact Lebesgue's work in other fields, the grandiosity of his achievements becomes even more overwhelming. One would need to write a whole book to give a fair presentation of all that he has accomplished. As far as the present short paper on the subject is concerned, let us end it by quoting the words of his friend and colleague Paul Montel, who has said about Lebesgue ([27], 18): "Il a été un grand savant, un professeur admirable, un homme d'une incomparable noblesse morale. Son influence sur le développement des mathématiques continuera longtemps à s'exercer par ses oeuvres propres et par celles qu'il a inspirées."

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