THE OLYMPIAD CORNER

No. 220

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We begin with the problems of the two days of the Turkish Mathematical Olympiad 1998. Thanks go to Ed Barbeau for collecting these problems at the IMO in Romania.

VI TURKISH MATHEMATICAL OLYMPIAD

Second Round

First Day - December 11, 1998 (Time: 4.5 hours)

1. On the base of the isosceles triangle ABC ($|AB| = |AC|$) we choose a point D such that $|BD| : |DC| = 2 : 1$ and on $[AD]$ we choose a point P such that $m(\widehat{BAC}) = m(\widehat{BPD})$.

Prove that $m(\widehat{DPC}) = m(\widehat{BAC})/2$.

2. Prove that

$$(a+3b)(b+4c)(c+2a) \;\geq\; 60abc$$

for all real numbers $0 \leq a \leq b \leq c$.

3. The points of a circle are coloured by three colours. Prove that there exist infinitely many isosceles triangles with vertices on the circle and of the same colour.

Second Day - December 12, 1998 (Time: 4.5 hours)

4. Determine all positive integers x, n satisfying the equation $x^3 + 3367 = 2^n$.

5. Given the angle XOY, variable points M and N are considered on the arms $[OX]$ and $[OY]$, respectively, so that $|OM| + |ON|$ is constant. Determine the geometric locus of the mid-point of $[MN]$.

6. Some of the vertices of unit squares of an $n \times n$ chessboard are coloured so that any $k \times k$ square formed by these unit squares on the chess board has a coloured point on at least one of its sides. If $l(n)$ stands for the minimum number of coloured points required to satisfy this condition, prove that

$$\lim_{n\to\infty} \frac{l(n)}{n^2} \;=\; \frac{2}{7}\,.$$

As a second set for this issue we give the Turkish Team Selection Examination for the 40^{th} IMO, 1999. Thanks again to Ed Barbeau for collecting them at the IMO in Romania.

TURKISH TEAM SELECTION EXAMINATION FOR THE 40^{th} IMO
First Day - March 20, 1999
(Time: 4.5 hours)

1. Let $m \leq n$ be positive integers and p be a prime. Let p–expansions of m and n be

$$\begin{aligned} m &= a_0 + a_1 p + \cdots + a_r p^r \,, \\ n &= b_0 + b_1 p + \cdots + b_s p^s \,, \end{aligned}$$

respectively, where a_r, $b_s \neq 0$, for all $i \in \{0, 1, \ldots, r\}$ and for all $j \in \{0, 1, \ldots, s\}$, we have $0 \leq a_i$, $b_j \leq p-1$.

If $a_i \leq b_i$ for all $i \in \{0, 1, \ldots, r\}$, we write $m \prec_p n$. Prove that

$$p \nmid \binom{n}{m} \quad \Longleftrightarrow \quad m \prec_p n \,.$$

2. Let L and N be the mid-points of the diagonals $[AC]$ and $[BD]$ of the cyclic quadrilateral $ABCD$, respectively. If BD is the bisector of the angle ANC, then prove that AC is the bisector of the angle BLD.

3. Determine all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that the set

$$\left\{ \frac{f(x)}{x} : x \neq 0 \quad \text{and} \quad x \in \mathbb{R} \right\}$$

is finite, and for all $x \in \mathbb{R}$

$$f(x - 1 - f(x)) \;=\; f(x) - x - 1 \,.$$

Second Day - March 21, 1999
(Time: 4.5 hours)

4. Let the area and the perimeter of a cyclic quadrilateral C be A_C and P_C, respectively. If the area and the perimeter of the quadrilateral which is tangent to the circumcircle of C at the vertices of C are A_T and P_T, respectively, prove that $\frac{A_C}{A_T} \geq \left(\frac{P_C}{P_T}\right)^2$.

5. Each of A, B, C, D, E and F knows a piece of gossip. They communicate by telephone via a central switchboard, which can connect only two of them at a time. During a conversation, each side tells the other everything he or she knows at that point. Determine the minimum number of calls for everyone to know all six pieces of gossip.

6. Prove that the plane is not a union of the inner regions of finitely many parabolas. (The outer region of a parabola is the union of the lines not intersecting the parabola. The inner region of a parabola is the set of points of the plane that do not belong to the outer region of the parabola.)

As a final contest for this issue we give the Final Round of the Japanese Mathematical Olympiad 1999. Thanks go to Ed Barbeau for collecting this when he was Canadian Team Leader to the IMO in Romania.

JAPANESE MATHEMATICAL OLYMPIAD 1999

Final Round — February 11, 1999

Duration: 4 hours

1. You can place a stone at each of 1999×1999 squares on a grid pattern. Find the minimum number of stones to satisfy the following condition.

Condition: When an arbitrary blank square is selected, the total number of stones placed in the corresponding row and column shall be 1999 or more.

2. Let $f(x) = x^3 + 17$. Prove that for each natural number n, $n \geq 2$, there is a natural number x, for which $f(x)$ is divisible by 3^n but not by 3^{n+1}.

3. Let $2n + 1$ weights (n is a natural number, $n \geq 1$) satisfy the following condition.

Condition: If any one weight is excluded, then the remaining $2n$ weights can be divided into a pair of n weights that balance each other.

Prove that all the weights are equal in this case.

4. Prove that

$$f(x) \;=\; (x^2+1^2)(x^2+2^2)(x^2+3^2)\cdots(x^2+n^2)+1$$

cannot be expressed as a product of two integral-coefficient polynomials with degree greater than 1.

5. For the convex hexagon $ABCDEF$ having side lengths that are all 1, find the maximum value M and minimum value m of three diagonals AD, BE, and CF and their possible ranges.

Next, we turn to readers' comments and solutions to problems of the 13$^{\text{th}}$ Iranian Mathematical Olympiad 1995, given [1999 : 456].

1. Find all real numbers $a_1 \leq a_2 \leq \cdots \leq a_n$ satisfying

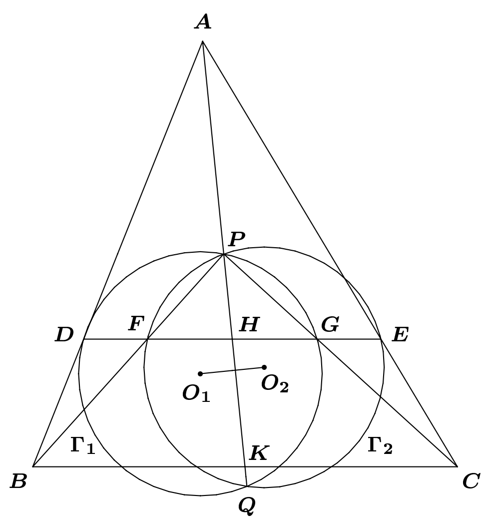
$$\sum_{i=1}^{n} a_i = 96, \quad \sum_{i=1}^{n} a_i^2 = 144, \quad \sum_{i=1}^{n} a_i^3 = 216.$$

Solutions by Moubinool Omarjee, Paris, France; and by George Evagelopoulos, Athens, Greece. Comments by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Pontoise, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Wang notes that this is the same as ***CRUX with MAYHEM*** problem #1982, which appeared [1994 : 250] with solution(s) [1995 : 256–257]. Bornsztein points out the similarity to ***CRUX with MAYHEM*** problem #1838, where a solution in positive integers is required.

2. Points D and E are situated on the sides AB and AC of triangle ABC in such a way that $DE \| BC$. Let P be an arbitrary point inside the triangle ABC. Lines PB and PC intersect DE at F and G, respectively. Let O_1 be the circumcentre of triangle PDG and let O_2 be that of PFE. Show that $AP \perp O_1O_2$.

Solution by Toshio Seimiya, Kawasaki, Japan.



Let Γ_1 and Γ_2 be the circumcircles of $\triangle PDG$ and $\triangle PFE$, respectively, so that O_1 and O_2 are centres of Γ_1 and Γ_2, respectively. Let Q be the intersection of Γ_1 and Γ_2 other than P. Then $O_1O_2 \perp PQ$.

Let H and K be the intersections of AP with DE and BC, respectively.

Since $DE \| BC$, we get

$$DH : BK = AH : AK = HE : KC \text{, so that}$$

$$DH : HE = BK : KC . \tag{1}$$

Since $FG \| BC$, we similarly get

$$FH : HG = BK : KC . \tag{2}$$

From (1) and (2), we have

$$DH : HE = FH : HG .$$

Thus,

$$DH \cdot HG = FH \cdot HE .$$

Note that $DH \cdot HG$ and $FH \cdot HE$ are the powers of H with respect to Γ_1 and Γ_2, respectively. Hence, H is a point on the radical axis of Γ_1 and Γ_2.

Since P and Q are intersections of Γ_1 and Γ_2, we have that PQ is the radical axis of Γ_1 and Γ_2. Therefore, H is a point on the line PQ, so that A, P, Q are collinear.

Since, $PQ \perp O_1O_2$, we have $AP \perp O_1O_2$.

3. Let $P(x)$ be a polynomial with rational coefficients such that $P^{-1}(\mathbb{Q}) \subseteq \mathbb{Q}$. Show that P is linear.

Solutions by Pierre Bornsztein, Pontoise, France; and by Moubinool Omarjee, Paris, France. We give Bornsztein's argument.

Let $P(x) \in \mathbb{Q}[x]$ such that $P^{-1}(\mathbb{Q}) \subset \mathbb{Q}$. It follows that

$$P(\mathbb{Q}) \subset \mathbb{Q} \text{ (since } P \in \mathbb{Q}[x]) \tag{1}$$

and

$$P(\mathbb{R} - \mathbb{Q}) \subset \mathbb{R} \backslash \mathbb{Q} \text{ (since } P^{-1}(\mathbb{Q}) \subset \mathbb{Q}) . \tag{2}$$

It is easy to see that P cannot be a constant polynomial.

Since $P \not\equiv 0$, multiply by the denominators of the coefficients of P. We obtain another polynomial with integer coefficients, satisfying (1) and (2).

Moreover, if c is the leading coefficient of this last polynomial, then, using $\left(x \mapsto \frac{x}{c}\right)$ and multiplying by c^{n-1} (where n is the degree of P), we obtain a monic polynomial, with (1) and (2).

Thus, with no loss of generality, we may suppose that $P \in \mathbb{Z}[x]$, where P has leading coefficient 1 (P is monic), and P satisfies (1) and (2).

The desired result follows immediately from this claim:

Claim: If $P \in \mathbb{Z}[x]$ is a monic polynomial, with degree greater than 1, then there exists an integer a such that $P(x)-a$ has a positive real irrational root.

Proof. Let p be a prime such that $p > P(1) - P(0)$ and greater than the largest real roots of $P(x) - P(0) - x$.

Let $a = p + P(0)$. Then $P(1) - a = P(1) - P(0) - p < 0$ and $P(p) - a = P(p) - P(0) - p > 0$.

From the Intermediate Value Theorem, it follows that $P(x) - a$ has a real root in $(1, p)$, say α.

Since $P(x)-a$ is a monic polynomial with integer coefficients, it is well known that, if α is a rational root of $P(x)-a$, then α divides $P(0)-a = -p$. Whence, $\alpha = 1$ or $\alpha = p$. Thus, α is irrational.

This ends the proof of the claim.

It follows from the claim that if $P \in \mathbb{Z}[x]$ is a monic polynomial satisfying (2), then P cannot have a degree greater than 1. That is, P is linear.

4. Let $S = \{x_1,\ x_2,\ \ldots,\ x_n\}$ be an n–element subset of the set $\{x \in R \mid x \geq 1\}$. Find the maximum number of elements of the form

$$\sum_{i=1}^{n} \varepsilon_i x_i\,, \quad \varepsilon_i = 0,\, 1$$

which belong to I, where I varies over all open intervals of length 1, and S over all n–element subsets.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Pontoise, France; and by Moubinool Omarjee, Paris, France. We give Bornsztein's argument.

We will prove that the desired maximum, denoted by M, is

$$M = \binom{n}{\lfloor \frac{n}{2} \rfloor} \text{ where } \lfloor\cdot\rfloor \text{ is the integer part function.}$$

Let S be an n–element subset of $(1, +\infty)$. Denote by $s(A)$ the sum of the elements of the subset A of S, with $s(\emptyset) = 0$. Let I be an open interval of length one.

Claim: If A, B are two subsets of S with $A \subsetneq B$ then $s(A)$ or $s(B)$ does not belong to I.

Proof. Since $A \subsetneq B$ there exists $x \in B\backslash A$. Thus,

$$s(B) \;=\; x + s(B - \{x\}) \;\geq\; x + s(A) \;\geq\; 1 + s(A)\,.$$

Then, $s(B) - s(A) \geq 1$.

It follows that the numbers $s(B)$ and $s(A)$ cannot belong to a common open interval of length one.

From the claim, it follows immediately that the number of elements of the form $\sum_{i=1}^{n} \varepsilon_i x_i$, with $\varepsilon_i = 0, 1$, (that is the number of $s(A)$ where $A \subset S$) which belong to I is not greater than the maximum number of subsets A_1, A_2, ..., of S which may be constructed such that none of the A_i's is included in another.

Such a family of subsets is a Sperner family. It is well known that a Sperner family has at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ elements (see [1]).

Since it is true for any I, S under the assumptions of the exercise, we deduce that

$$M \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}. \qquad (1)$$

Conversely: Let p be an integer such that $\lfloor \frac{n}{2} \rfloor < p$.

For $i = 1, \ldots, n$, let $x_i = 1 + \frac{1}{p+i}$.

Let A be any $\left\lfloor \frac{n}{2} \right\rfloor$–subset of $S = \{x_1, \ldots, x_n\}$. Then,

$$\begin{aligned}
\left\lfloor \frac{n}{2} \right\rfloor < s(A) &= \left\lfloor \frac{n}{2} \right\rfloor + \sum_{\substack{i \\ x_i \in A}} \frac{1}{p+i} \\
&< \left\lfloor \frac{n}{2} \right\rfloor + \sum_{\substack{i \\ x_i \in A}} \frac{1}{p} \\
&= \left\lfloor \frac{n}{2} \right\rfloor + \frac{1}{p} \left\lfloor \frac{n}{2} \right\rfloor .
\end{aligned}$$

Thus,

$$s(A) \in I = \left(\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) .$$

I is clearly an open interval of length one, and I contains all $s(A)$ where $A \subset S$ with $\mathbf{Card}(A) = \lfloor \frac{n}{2} \rfloor$.

Then, at least $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ elements of the form $\sum \varepsilon_i x_i$, $\varepsilon_i = 0, 1$, belong to I.

It follows that

$$M \geq \binom{n}{\lfloor \frac{n}{2} \rfloor}. \qquad (2)$$

From (1) and (2), we deduce that $M = \binom{n}{\lfloor \frac{n}{2} \rfloor}$, as claimed.

Reference.

[1] K. Engel, "Sperner theory", Cambridge University Press, p. 1–3.

5. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ that fulfils all of the following conditions:

(a) $f(1) = 1$

(b) there exists $M > 0$ such that $-M < f(x) < M$

(c) if $x \neq 0$ then

$$f\left(x + \frac{1}{x^2}\right) \;=\; f(x) + \left(f\left(\frac{1}{x}\right)\right)^2 \text{ ?}$$

Solution by Mohammed Aassila, Strasbourg, France.

Let n be the smallest integer for which $f(x) < n$ for all $x \neq 0$. Then, we can find $x \neq 0$ such that $f(x) \geq n - 1$. Then,

$$\left(f\left(\frac{1}{x}\right)\right)^2 \;=\; f\left(x + \frac{1}{x^2}\right) - f(x) \;<\; n - (n-1) \;=\; 1\,,$$

and thus, $f(\frac{1}{x}) > -1$. Now, substituting $\frac{1}{x}$ for x in the original equation, we have

$$(n-1)^2 \;\leq\; f(x)^2 \;=\; f\left(\frac{1}{x^2} + x\right) - f\left(\frac{1}{x}\right) \;<\; n + 1\,.$$

Thus, $(n-1)^2 < n + 1$, and thus, $n \in \{1,\, 2\}$. But, putting $x = 1$ in the original equation, we get $f(2) = 2$, and therefore, $n > 2$, a contradiction.

Comment by Pierre Bornsztein, Pontoise, France. The problem and solution can be found in "*36*th *International Mathematical Olympiad*" published by the Canadian Mathematical Society, p. 124.

6. Let a, b, c be positive real numbers. Find all real numbers x, y, z such that

$$x + y + z \;=\; a + b + c$$
$$4xyz - (a^2x + b^2y + c^2z) \;=\; abc\,.$$

Solution by Mohammed Aassila, Strasbourg, France.

The second equation is equivalent to

$$4 \;=\; \frac{a^2}{yz} + \frac{b^2}{zx} + \frac{c^2}{xy} + \frac{abc}{xyz}$$

and also to

$$4 \;=\; x_1^2 + y_1^2 + z_1^2 + x_1y_1z_1\,,$$

where

$$0 \;<\; x_1 \;=\; \frac{a}{\sqrt{yz}} \;<\; 2\,, \qquad 0 \;<\; y_1 \;=\; \frac{b}{\sqrt{zx}} \;<\; 2\,,$$
$$0 \;<\; z_1 \;=\; \frac{c}{\sqrt{xy}} \;<\; 2\,.$$

Setting $x_1 = 2\sin u$, $0 < u < \frac{\pi}{2}$, and $y_1 = 2\sin v$, $0 < v < \frac{\pi}{2}$, we have

$$4 \;=\; 4\sin^2 u + 4\sin^2 v + z_1^2 + 4\sin u \cdot \sin v \cdot z_1\,.$$

Hence,

$$z_1 + 2\sin u \cdot \sin v \;=\; 2\cos u \cdot \cos v\,,$$

and then,

$$z_1 \;=\; 2(\cos u \cdot \cos v - \sin u \cdot \sin v) \quad \big(=\; 2\cos(u+v)\big)\,.$$

Thus,

$$\begin{aligned} a &= 2\sqrt{yz}\sin u\,, \\ b &= 2\sqrt{zx}\sin v\,, \\ c &= 2\sqrt{xy}(\cos u \cdot \cos v - \sin u \cdot \sin v)\,. \end{aligned}$$

From $x+y+z = a+b+c$, we get

$$\big(\sqrt{x}\cos v - \sqrt{y}\cos u\big)^2 + \big(\sqrt{x}\sin v + \sqrt{y}\sin u - \sqrt{z}\big)^2 \;=\; 0$$

which implies

$$\sqrt{z} \;=\; \sqrt{x}\sin v + \sqrt{y}\sin u \;=\; \sqrt{x}\,\frac{y_1}{2} + \sqrt{y}\,\frac{x_1}{2}\,.$$

Therefore,

$$\sqrt{z} \;=\; \sqrt{x}\cdot\frac{b}{2\sqrt{zx}} + \sqrt{y}\cdot\frac{a}{2\sqrt{yz}}\,,$$

and thus, $z = \dfrac{a+b}{2}$. Similarly, $x = \dfrac{a+b}{2}$, $y = \dfrac{c+a}{2}$.

The triple

$$(x,y,z) \;=\; \left(\frac{b+c}{2}, \frac{c+a}{2}, \frac{a+b}{2}\right)$$

is the unique solution.

Comment by Pierre Bornsztein, Pontoise, France. The problem and a solution are in "*36th International Mathematical Olympiad*", published by the Canadian Mathematical Society, p. 122–123.

Next we turn to the February 2000 number of the *Corner* and solutions to problems of the Final (Selection) Round of the Estonian Mathematical Contests 1995-96 given [2000 : 6].

1. The numbers x, y and $\dfrac{x^2+y^2+6}{xy}$ are positive integers. Prove that $\dfrac{x^2+y^2+6}{xy}$ is a perfect cube.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's write-up and comment.

Suppose $\dfrac{x^2+y^2+6}{xy} = k$ for some positive integers x, y and k. We prove that necessarily $k = 8$.

Consider

$$\frac{x^2+y^2+6}{xy} = k \tag{1}$$

as a Diophantine equation in two variables x and y. Let (a,b) denote the solution of (1) with the *least* positive a value. We first show that $a = 1$. Due to symmetry, we may assume that $a \le b$. Note first that a and b must both be odd since they clearly must have the same parity and if they are both even, then modulo 4, $a^2+b^2+6 \equiv 2$ while $kab \equiv 0$. If $a = b$, then from $k = 2 + \frac{6}{a^2}$ we deduce immediately that $a = 1$. If $a < b$ then $b \ge a+1$.

Since

$$\frac{(ka-b)^2+a^2+6}{(ka-b)a} = \frac{k^2a^2-2kab+kab}{ka^2-ab} = k$$

and

$$ka-b = \frac{a^2+b^2+6}{b} - b = \frac{a^2+6}{b} > 0\,,$$

we see that $(ka-b,a)$ is also a solution of (1) in natural numbers. Note that $a^2+6-ab \le a^2+6-a(a+1) = 6-a < 0$ if $a > 6$. Thus, $ka-b = \frac{a^2+6}{b} < a$ if $a > 6$, contradicting the minimality of a. Hence, $a \le 6$. It remains to show that $a \ne 3$, 5. If $a = 3$, then we get $b^2+15 = 3kb$, which implies that 3 divides b and thus, $b^2+15 \equiv 6$, while $3kb \equiv 0 \pmod 9$. If $a = 5$, then we get $b^2+31 = 5kb$. Since b is an integer and 31 is a prime we deduce from the relation between roots and coefficients that $5k = 32$, which is impossible. Therefore, we conclude that $a = 1$, which implies $k = \dfrac{b^2+7}{b} = b + \dfrac{7}{b}$. Hence, $b = 7$ and $k = 8$, which is a cube.

Comment. This is a beautiful problem which parallels the following "infamous" problem of the 29th IMO, supposedly the most difficult IMO problem ever (with an average score of 0.6 out of 7):

> "Suppose a and b are positive integers such that $ab+1$ divides a^2+b^2. Prove that $\dfrac{a^2+b^2}{ab+1}$ is a perfect square."

2. Let a, b, c be the sides of a triangle and α, β, γ the opposite angles of the sides, respectively. Prove that if the inradius of the triangle is r, then $a\sin\alpha + b\sin\beta + c\sin\gamma \geq 9r$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea; by David Loeffler, student, Cotham School, Bristol, UK; by Toshio Seimiya, Kawasaki, Japan; and by Panos E. Tsaoussoglou, Athens, Greece. We first give Loeffler's solution.

First we substitute $\sin(\alpha) = \frac{a}{2R}$, etc., so that the required inequality is equivalent to

$$9r \;\leq\; \frac{a^2}{2R} + \frac{b^2}{2R} + \frac{c^2}{2R}\,, \qquad \text{or} \qquad 18Rr \;\leq\; a^2 + b^2 + c^2\,.$$

The product Rr can be neatly expressed by comparing two well-known formulae for the area of the triangle: $\frac{abc}{4R}$ and $rs = \frac{r(a+b+c)}{2}$. Equating these gives $Rr = \frac{abc}{2(a+b+c)}$.

Thus, the required inequality becomes:

$$\frac{9abc}{a+b+c} \;\leq\; a^2 + b^2 + c^2\,,$$

$$9abc \;\leq\; (a^2 + b^2 + c^2)(a+b+c)\,.$$

However, applying the AM-GM inequality, we see that

$$(abc)^{\frac{2}{3}} \;\leq\; \frac{a^2 + b^2 + c^2}{3}\,;$$

that is,

$$3(abc)^{\frac{2}{3}} \;\leq\; a^2 + b^2 + c^2\,,$$

and

$$(abc)^{\frac{1}{3}} \;\leq\; \frac{a+b+c}{3}\,, \qquad \text{or} \qquad 3(abc)^{\frac{1}{3}} \;\leq\; a+b+c\,.$$

Multiplying these together it follows that $9abc \leq (a+b+c)(a^2+b^2+c^2)$, which we showed to be equivalent to the required inequality.

Now we give Klamkin's generalization and comment.

Since $a = 2R\sin\alpha$, etc., where R is the circumradius, the inequality is now

$$a^2 + b^2 + c^2 \;\geq\; 18Rr\,.$$

We prove the stronger inequality

$$a^2 + b^2 + c^2 \;\geq\; xRr - (2x - 36)r^2 \;\geq\; 18Rr\,,$$

where $18 \leq x \leq 24$. The right-hand inequality reduces to $(x-18)(R-2r) \geq 0$

which is the well-known Chapple-Euler inequality. For the left-hand inequality, we use an identity [1; p. 52].

$$a^2+b^2+c^2 \;=\; 2(s^2-4Rr-r^2) \;\geq\; xRr-(2x-36)r^2$$

where s is the semiperimeter. This reduces the inequality to

$$2s^2 \;\geq\; (x+8)Rr+(38-2x)r^2\ .$$

Since it is known [1; p. 50] that $s^2 \geq 16Rr-5r$, we must have here that

$$32Rr-10r^2 \;\geq\; (x+8)Rr+(38-2x)r^2$$

or that $(24-x)Rr \geq (48-2x)r^2$ and which is valid if $x \leq 24$ as well.

Comment. As a known complementary inequality, we also have

$$9R^2 \;\geq\; a^2+b^2+c^2\ .$$

For one simple proof of this consider the expansion of $(A+B+C)^2 \;\geq\; 0$ where A, B, C are vectors from the circumcentre to the vertices A, B, C of the triangle ABC (note that $A^2=R^2$, $2B\cdot C=2R^2-a^2$, etc.). Another proof follows from $OH^2 = 9R^2-(a^2+b^2+c^2)$ where O and H are the circumcentre and orthocentre of ABC.

Reference:
[1] D.S. Mitrinovic, J.E. Pecaric, V. Volenic, **Recent Advances in Geometric Inequalities**, Kluwer, Dordrecht, 1989.

3. Prove that the polynomial $P_n(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots+\frac{x^n}{n!}$ has no zeros if n is even and has exactly one zero if n is odd.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give the write-up of Bataille (with his added supplementary remark).

We shall prove by induction the following property

$$(\pi_n)\begin{cases} P_{2n} & \text{has no (real) zeros} \\ P_{2n+1} & \text{has exactly one (real) zero.}\end{cases}$$

Clearly (π_0) is true since $P_0(x)=1$ and $P_1(x)=1+x$.

Suppose now that (π_n) is true for an integer $n\geq 0$. We will denote by x_n the unique zero of P_{2n+1}. The continuous function P_{2n} has no zeros and is positive for $x\geq 0$. Hence, $P_{2n}(x)>0$ for all x.

Since the derivative P'_{2n+1} of P_{2n+1} is P_{2n}, the function P_{2n+1} is increasing. Furthermore, $\lim\limits_{x\to-\infty} P_{2n+1}(x)=-\infty$ and $\lim\limits_{x\to+\infty} P_{2n+1}(x)=+\infty$, so that $P_{2n+1}(x)<0$ for $x<x_n$ and $P_{2n+1}(x)>0$ for $x>x_n$.

Now, using $P'_{2n+2} = P_{2n+1}$, we see that $P_{2n+2}(x)$ is a minimum when $x = x_n$. Hence, for all x

$$\begin{aligned} P_{2n+2}(x) \ \geq \ P_{2n+2}(x_n) &= P_{2n+1}(x_n) + \frac{x_n^{2n+2}}{(2n+2)!} \\ &= \frac{x_n^{2n+2}}{(2n+2)!} = \frac{(x_n^{n+1})^2}{(2n+2)!} > 0\,. \end{aligned}$$

Thus, $P_{2n+2}(x)$ never takes the value 0.

Also, as above, P_{2n+3} is increasing, continuous, and $\lim\limits_{x\to-\infty} P_{2n+3}(x) = -\infty$, $\lim\limits_{x\to+\infty} P_{2n+3}(x) = +\infty$. Hence, P_{2n+2} has a unique zero. This completes the induction and shows that property (π_n) is true for all non-negative integer n.

Remark. As a supplement, we show that $\lim\limits_{n\to\infty} x_n = -\infty$.

Consider

$$P_{2n+1}(-2n-3) = \sum_{k=0}^{2n+1} \frac{(-2n-3)^k}{k!} = \sum_{p=0}^{n} (2n+3)^{2p} \left(\frac{1}{(2p)!} - \frac{2n+3}{(2p+1)!} \right) .$$

We have $P_{2n+1}(-2n-3) < 0$ (since $2n+3 > 2p+1$ for $p = 0, \dots, n$) and thus, $x_n > -2n-3$. Hence,

$$P_{2n+3}(x_n) = P_{2n+3}(x_n) - P_{2n+1}(x_n) = \frac{x_n^{2n+2}}{(2n+2)!} \left(1 + \frac{x_n}{2n+3} \right) > 0 \ ,$$

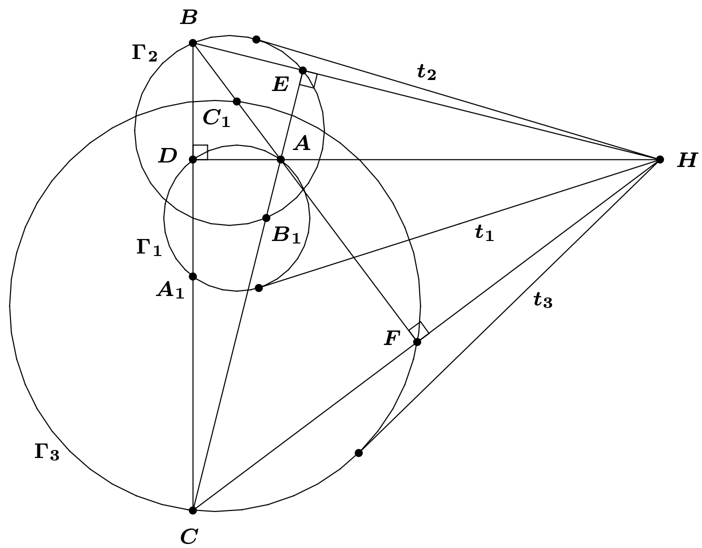
which implies $x_n > x_{n+1}$. Therefore, (x_n) is a decreasing sequence of real numbers and, as such, either $\lim_{n\to\infty} x_n = -\infty$ or (x_n) converges to a real number m. Assume that the latter does occur. Note that $m \leq x_n < 0$ for all n. Since for all $n \geq 0$ we have $P_{2n+1}(x) \leq e^x \leq P_{2n}(x)$ for all $x \leq 0$ [easy induction], we would have

$$\begin{aligned} 0 \ \leq \ e^{x_n} \ \leq \ P_{2n}(x_n) &= P_{2n+1}(x_n) - \frac{x_n^{2n+1}}{(2n+1)!} \\ &= -\frac{x_n^{2n+1}}{(2n+1)!} \leq -\frac{m^{2n+1}}{(2n+1)!}\,. \end{aligned}$$

But then, $\lim_{n\to\infty} e^{x_n} = 0$ (because $\lim\limits_{n\to\infty} \dfrac{m^{2n+1}}{(2n+1)!} = 0$) while, by the continuity of the exponential function, we must have $\lim\limits_{n\to\infty} e^{x_n} = e^m \neq 0$. This contradiction shows that $\lim\limits_{n\to\infty} x_n = -\infty$.

4. Let H be the orthocentre of an obtuse triangle ABC and A_1, B_1, C_1 arbitrary points taken on the sides BC, AC, AB, respectively. Prove that the tangents drawn from the point H to the circles with diameters AA_1, BB_1, CC_1 are equal.

Solutions by Michel Bataille, Rouen, France; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.



The lines HA, HB, HC meet BC, CA, AB at D, E, F, respectively.

Then, $AD \perp BC$, $BE \perp AC$ and $CF \perp AB$.

Let Γ_1, Γ_2, Γ_3 be circles with diameters AA_1, BB_1, CC_1, respectively.

Since $\angle ADA_1 = \angle BEB_1 = CFC_1 = 90^\circ$, we have that Γ_1, Γ_2, Γ_3 pass through D, E, F, respectively.

Let t_1, t_2, t_3 be the tangent segments from H to Γ_1, Γ_2, Γ_3, respectively.

Since $\angle ADB = \angle AEB = 90^\circ$, A, D, B, E are concyclic, so that

$$HA \cdot HD \;=\; HB \cdot HE\,.$$

Hence, we have

$$t_1^2 \;=\; HA \cdot HD \;=\; HB \cdot HE \;=\; t_2^2\,.$$

Thus, $t_1 = t_2$.

Similarly, we have $t_1 = t_3$. Therefore, $t_1 = t_2 = t_3$.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions for all $x \in \mathbb{R}$.

(a) $f(x) = -f(-x)$;

(b) $f(x+1) = f(x) + 1$;

(c) $f\left(\frac{1}{x}\right) = \frac{1}{x^2} f(x)$, if $x \neq 0$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give the write-up by Aassila.

Set $g(x) = f(x) - x$. Then g satisfies

(a') $g(x) = -g(-x)$;

(b') $g(x+1) = g(x)$;

(c') $g\left(\frac{1}{x}\right) = \frac{1}{x^2} g(x)$, if $x \neq 0$.

By (a')–(b') we get $g(0) = g(-1) = 0$, and for all $x \neq 0$, -1 we have

$$\begin{aligned} g(x) &= g(x+1) \;=\; (x+1)^2 g\left(\frac{1}{x+1}\right) \;=\; -(x+1)^2 g\left(-\frac{1}{x+1}\right) \\ &= -(x+1)^2 g\left(1 - \frac{1}{x+1}\right) \;=\; -(x+1)^2 g\left(\frac{x}{x+1}\right) \\ &= -(x+1)^2 \frac{x^2}{(x+1)^2} g\left(\frac{x+1}{x}\right) \;=\; -x^2 g\left(1 + \frac{1}{x}\right) \;=\; -x^2 g\left(\frac{1}{x}\right) \\ &= -g(x)\,. \end{aligned}$$

Hence, $g(x) \equiv 0$, and $f(x) = x$ for all $x \in \mathbb{R}$.

Now we turn to solutions from our readers to problems of the Japan Mathematical Olympiad, Final Round, 1996 given [2000 : 7].

2. For positive integers m, n with $\gcd(m,n) = 1$, determine the value $\gcd(5^m + 7^m, 5^n + 7^n)$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by George Evagelopoulos, Athens, Greece. We give Bornsztein's solution.

We will use the following lemma (see [1] for a proof).

Lemma. If a and b are relatively prime integers with $a > b$, then for every pair of positive integers m and n we have

$$(a^m - b^m, a^n - b^n) \;=\; a^{(m,n)} - b^{(m,n)}$$

where (x,y) denotes the gcd of integers x and y.

Let m, n be positive integers, with $(m,n) = 1$.

Let $S_m = 5^m + 7^m$, $S_n = 5^n + 7^n$, $\mathcal{S} = (S_m, S_n)$ and $U_a = 7^a - 5^a$ for $a \in \mathbb{N}^*$.

If $m = n = 1$, we have $\mathcal{S} = S_m = S_n = 12$.

Since $(m,n) = 1$ and m, n are playing symmetric parts, we may suppose that $m > n$.

Let $m = a + n$. Then, $a \in \mathbb{N}^*$ and $(a, n) = 1$.

We have

$$S_m - 5^a S_n \;=\; 7^n U_a\,. \tag{1}$$

Since 5 and 7 are relatively prime, it is easy to see that $(\mathcal{S}, 5^a) = 1$ and $(\mathcal{S}, 7^n) = 1$.

Then, from (1), we deduce that $\mathcal{S} = (U_a, S_n)$.

Case 1. a is odd.

Let $S_n = \mathcal{S}l$, $U_a = \mathcal{S}k$, where k, $l \in \mathbb{N}^*$ and $(k, l) = 1$. Then, using the binomial expansion

$$7^{an} \;=\; (\mathcal{S}l - 5^n)^a \;=\; \mathcal{S}L - 5^{an}$$

and

$$7^{an} \;=\; (\mathcal{S}k + 5^a)^n \;=\; \mathcal{S}K + 5^{an}$$

where K, L are integers.

Thus,

$$\mathcal{S}(K - L) \;=\; 2 \cdot 5^{an}\,.$$

It follows that $\mathcal{S}$ divides $2 \cdot 5^{an}$. Since $(\mathcal{S}, 5) = 1$, we deduce (using Gauss' theorem) that $\mathcal{S}$ divides 2.

Moreover, S_m, S_n are even. Thus, 2 divides $\mathcal{S}$. Then, $\mathcal{S} = 2$.

Case 2. a is even.

Let $a = 2b$, where $b \in \mathbb{N}^*$ and $(b, n) = 1$.

Moreover, m and n are odd, since $m = 2b + n$ and $(m, n) = 1$. Then $\mathcal{S}$ divides $S_n U_n = U_{2n} = 7^{2n} - 5^{2n}$ and $\mathcal{S}$ divides $U_a = 7^{2b} - 5^{2b}$.

From the lemma, we deduce that $\mathcal{S}$ divides

$$7^2 - 5^2 = 24\,. \tag{2}$$

Since m, n are odd, we have $7^m \equiv -1 \pmod 8$ and $5^m \equiv 5 \pmod 8$.

It follows that 4 divides S_m and S_n, but 8 does not divide either S_m or S_n.

In the same way, $S_m \equiv S_n \equiv 1^m + (-1)^m \equiv 0 \pmod 3$. Since 3 and 4 are relatively prime, it follows that

$$S_m \;\equiv\; S_n \;\equiv\; 0 \pmod{12}$$

but

$$S_m \;\not\equiv\; 0 \pmod{24}\;.$$

From (2), we may conclude that $\mathcal{S} = 12$. Then:

if m, n are odd, we have $(S_m, S_n) = 12$;

if m, n have opposite parities, we have $(S_m, S_n) = 2$.

Reference.

[1] *Math. Magazine*, Exercise no 1091, vol. 54, no. 2, March 1981, pp. 86–87.

3. Let x be a real number with $x > 1$ and such that x is not an integer. Let $a_n = \lfloor x^{n+1} \rfloor - x\lfloor x^n \rfloor$ $(n = 1, 2, 3, \ldots)$. Prove that the sequence of numbers $\{a_n\}$ is not periodic. (Here $\lfloor y \rfloor$ denotes, as usual, the largest integer $\leq y$.)

Solutions by Mohammed Aassila, Strasbourg, France; and by Michel Bataille, Rouen, France. We give Bataille's solution.

Suppose for purpose of contradiction, that there exists a positive integer p such that $a_{n+p} = a_n$ for all positive integers n. Defining the integer u_n by $u_n = \lfloor x^{n+p} \rfloor - \lfloor x^n \rfloor$, we would have $u_{n+1} = xu_n$ for all n. We now distinguish the following mutually exclusive cases:

Case 1. $u_1 = \lfloor x^{p+1} \rfloor - \lfloor x \rfloor \neq 0$.

Then, $x = \frac{u_2}{u_1}$ is a rational number that we can also write as $x = \frac{k}{l}$ where k, l are coprime integers. Note that $k > l > 1$ (since $x > 1$ and $x \notin \mathbb{N}$). For all n, $u_{n+1} = x^n u_1$ so that $l^n u_{n+1} = k^n u_1$ and, l^n being coprime with k^n, l^n divides u_1. This would mean that u_1 has an infinite number of divisors (since $l > 1$), which is clearly impossible.

Case 2. $u_1 = \lfloor x^{p+1} \rfloor - \lfloor x \rfloor = 0$.

Then, $u_n = 0$ for all n, and an easy induction shows that, for all positive integers m,

$$\left.\begin{array}{c} \lfloor x^{mp+1} \rfloor \;=\; \lfloor x \rfloor\,,\, \lfloor x^{mp+2} \rfloor \;=\; \lfloor x^2 \rfloor\,, \ldots,\, \lfloor x^{mp+p-1} \rfloor \;=\; \lfloor x^{p-1} \rfloor \\ \text{and} \quad \lfloor x^{mp} \rfloor \;=\; \lfloor x^p \rfloor\,. \end{array}\right\} \qquad (1)$$

However, from $x < x^2 < \cdots < x^p < x^{p+1}$, we get

$$\lfloor x \rfloor \leq \lfloor x^2 \rfloor \leq \cdots \leq \lfloor x^p \rfloor \leq \lfloor x^{p+1} \rfloor$$

and since $\lfloor x^{p+1} \rfloor = \lfloor x \rfloor$, we have $\lfloor x^r \rfloor = \lfloor x \rfloor$ for $r = 1, 2, \ldots, p+1$. Actually, we even have $\lfloor x^n \rfloor = \lfloor x \rfloor$ for all positive integers n (divide n by p and use (1)). Again this is impossible since $\lim\limits_{n\to\infty} x^n = +\infty$ and thus, $\lim\limits_{n\to\infty} \lfloor x^n \rfloor = +\infty$.

Thus, in both cases we are led to an impossibility, and $\{a_n\}$ cannot be periodic.

5. Let q be a real number such that $\frac{1+\sqrt{5}}{2} < q < 2$. When we represent a positive integer n in binary expansion as

$$n = 2^k + a_{k-1} \cdot 2^{k-1} + \cdots + a_1 \cdot 2 + a_0$$

(here $a_i = 0$ or 1), we define p_n by

$$p_n = q^k + a_{k-1} q^{k-1} + \cdots + a_1 q + a_0 .$$

Prove that there exist infinitely many positive integers k which satisfy the following condition: There exists no positive integer l such that $p_{2k} < p_l < p_{2k+1}$.

Solution by Mohammed Aassila, Strasbourg, France.

By induction on n, we can prove that $k = q_n$ satisfies the required condition, where q_n is defined by

$$\begin{cases} q_{2m} & = \sum\limits_{k=0}^{m} 2^{2k} \\ q_{2m+1} & = \sum\limits_{k=0}^{m} 2^{2k+1} . \end{cases}$$

That completes the *Corner* for this issue of ***CRUX with MAYHEM***. We are entering Olympiad season. Send me your nice solutions and generalizations as well as Olympiad Contests.

Professor Toshio Seimiya

Regular readers of this section will be aware of the many beautiful geometry problems that have been proposed by Professor Toshio Seimiya. We dedicated some to him in March 2001 [2001 : 114], in celebration of his 90^{th} birthday. Unfortunately, we did not have a photograph available then. We do now! See page 100.

BOOK REVIEWS

JOHN GRANT McLOUGHLIN

Teaching Statistics: Resources for Undergraduate Instructors
edited by Thomas L. Moore, published by the Mathematical Association of America (MAA Notes Series, #52), 2000,
ISBN 0-88385-162-8, softcover, 222+xii pages, $31.95 (U.S.).
Reviewed by ***C.L. Kaller****, retired Professor of Mathematics, Kelowna, BC.*

This instructors' handbook is an eclectic collection of articles (some published for the first time, but many reprints of previously published articles or commentaries on such articles) on a number of aspects of introductory undergraduate statistics instruction. It is a compendium of information intended to be regarded essentially as an instructors' manual for teachers of statistics courses to undergraduate or even senior secondary school students. Particular targets of this volume are those teachers of statistics courses who have limited formal training in that discipline.

The volume consists of articles in six categories:

1. Hortatory Imperatives [of data-based statistical instruction]
2. Teaching with Data [in classroom settings]
3. Established Projects in Active Learning [with usage guidelines]
4. Textbooks [with detailed textbook selection procedures]
5. Technology [resources available for classroom presentations]
6. Assessment [of student achievement].

The motivation behind the choice of articles in each of these categories is relevance to what the editor feels is the current (2000) thinking about just what constitutes acceptable statistics instruction, particularly when such instruction is provided by teachers whose formal professional training in statistics is somewhat limited. The editor points out that the volume is far from an all-inclusive presentation of resources and ideas on statistics education; it is his stated hope, however, that the material will provide direction to readers and to have them keep alert to other instructional resources constantly being developed.

Laudatory as is the intent of the editor in publishing the contents of this handbook to help the classroom statistics instructor, this publication is not without some irritating editorial flaws giving this reviewer the distinct impression that the volume was thrown together in a hasty rush to the printers. Grammar, spelling, consistency and format have not received the careful editorial attention expected in an MAA publication. Indeed, on the back cover of the book (as well as elsewhere inside) even the word *statistics* is misspelled!

Very irritating is the sloppiness in grammar, especially in the use of the word *data* which appears inconsistently presented as both a plural and a singular noun. Indeed, within the same paragraph in the volume, this word appears as a singular noun in one sentence and plural in another, and authors inconsistently refer to *these data* and *this data* quite interchangeably. Careful editing of the articles accepted for this publication would at least have resulted in consistently using (incorrectly, asserts this reviewer) the word *data* in singular form if it were impossible to use it correctly as a plural noun.

In the editor's own contribution (on page 31, *Implications for Statistics Teaching*) it is stated that "Statistics faculty need to learn new skills. We now have to grade grammar and writing style as well as statistical thinking." In a volume intended to give advice and guidance to statistics teachers, it is regrettable that the editor/author failed to take his own pronouncement to heart to produce a better quality contribution to the *MAA Notes Series*.

Gardner's Workout: Training the Mind and Entertaining the Spirit
by Martin Gardner, published by A.K. Peters, 2001.
ISBN 1-56881-120-9, hardcover, 319 + xi pages, $35.00 (U.S.).
Reviewed by **Edward J. Barbeau**, *University of Toronto, Toronto, Ontario.*

Martin Gardner has been part of my mathematical environment since I discovered his *Scientific American* columns as an undergraduate and was captivated by his easy style and elegant problems. Even though he has been succeeded by others in *Scientific American*, he has remained very much in view through appearances in television specials, issues and reissues of his essays, continuing interest of others in problems that he was the first to publicize and articles in a number of journals.

This book is an anthology of 34 articles and 7 reviews that have appeared, mostly during the 1990s, in *Quantum*, *Math Horizons* and other journals of general mathematical interest. An article on Kasparov's defeat by *Deep Blue* first appeared in *The Washington Post*; four reviews of books were published in *The Los Angeles Times*; an essay on the growth of recreational mathematics appeared in *Scientific American*, although that was a "heavily revised and cut version" of what appears in this book. Despite the variety of topics, there is a linkage from many essays to the next that provides a coherent flow of ideas for the reader who, like myself, progressed from end to end. Prominent are articles on number play, geometry, graph theory, artificial intelligence, puzzles, games and tricks.

Occasionally Gardner becomes more serious. In a pair of articles on artificial intelligence, Gardner dissents from the views of those who would see us on the threshold of reproducing human thinking, and puts forward his belief that the mind is on a much higher plan of complexity and is qualitatively different from anything that has been produced so far on computers. A review of the 1997 *NCTM Yearbook*, an algebra text and the PBS videoseries *Life by the Numbers* in *The New York Review of Books* provides the occasion to vent his displeasure at the "fuzzy new math". He skewers it for its faddishness and dilution with much that is mathematically irrelevant. If the *Yearbook* is any guide, reformers seem to be completely ignorant of fascinating and challenging mathematical material available in a number of fine books published in recent years by established mathematicians. It is hard to argue with his assessment of the PBS series as "high on special effects, low on mathematical content". This is an important article, that well deserves its more permanent place in this volume.

Those who want to enliven the modern school classroom will find many riches in this book. Four successive chapters treat, in whole or in part, magic squares, with some new material. A prize of $100 has not yet been claimed for a 3×3 magic square whose entries are all squares. There are many squares that almost do the job, but getting the complete solution is equivalent to finding rational points on certain cubic curves. There are numerous dissection problems, a genre which I have found encourages my own undergraduate students to think of geometry in more structural terms. A *Quantum* article deals with decomposing both squares and equilateral triangles into three similar parts, all three of which, two of which and none of which are congruent. Is it true that there is a unique solution to the triangle problems with one and no congruent pairs of parts? In the magazine, *Cubism for Fun*, Gardner offered a $50 prize for anyone who could, for any integer n exceeding 1, cover the surface of a cube by n congruent polygons without overlapping. In an addendum, he tells us that this was won by a reader, Anneke Treep, who without any technical complications applied the right perspective and degree of imagination.

Other problems, however, are definitely not for the classroom. Consider the problem, due to Gardner himself from the initial chapter of the book; it asks for the minimum area of a surface placed inside a transparent cube to render it opaque from any direction. The best answer seems to be 4.2324. "I believe the opaque cube problem to be extremely difficult," he writes, "it is keeping me awake at nights."A substantial essay treats minimal Steiner trees (spanning trees of minimum length) on a rectangular array of nodes, and includes a table of the best known results for $n \times n$ when $2 \leq n \leq 14$.

The pace is varied by a discussion of directed graphs to analyze propositional calculus, and by some word play. Can you provide a square array of nine letters for which, each row, column and diagonal, spell out a word? Or provide a chain of words, each altered by a single letter from its predecessor, that converts BLACK to WHITE?

The effectiveness of this book derives in large part from the passion with which Gardner shares his mathematical enthusiasm and on the breadth and erudition of his discussions. This is another winner!

On a "Problem of the Month"

Murray S. Klamkin

In the problem of the month [1999 : 106], one was to prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \;\le\; \sqrt{a}+\sqrt{b}+\sqrt{c}\,,$$

where a, b, c are sides of a triangle.

It is to be noted that this inequality will follow immediately from the Majorization Inequality [1]. Here, if $\mathbf{A}$ and $\mathbf{B}$ are vectors $(a_1, a_2, \ldots, a_n)$, $(b_1, b_2, \ldots, b_n)$ where $a_1 \ge a_2 \ge \cdots \ge a_n$, $b_1 \ge b_2 \ge \cdots \ge b_n$, and $a_1 \ge b_1$, $a_1+a_2 \ge b_1+b_2$, $\ldots$, $a_1+a_2+\cdots+a_{n-1} \ge b_1+b_2+\cdots+b_{n-1}$, $a_1+a_2+\cdots+a_n = b_1+b_2+\cdots+b_n$, we say that $\mathbf{A}$ majorizes $\mathbf{B}$ and write it as $\mathbf{A} \succ \mathbf{B}$. Then, if F is a convex function,

$$F(a_1)+F(a_2)+\cdots+F(a_n) \;\ge\; F(b_1)+F(b_2)+\cdots+F(b_n)\,.$$

If F is concave, the inequality is reversed.

For the triangle inequality, we can assume without loss of generality that $a \ge b \ge c$. Then $a+b-c \ge a$, $(a+b-c)+(a+c-b) \ge a+b$, and $(a+b-c)+(a+c-b)+(b+c-a) = a+b+c$. Therefore, if F is concave,

$$F(a+b-c)+F(b+c-a)+F(c+a-b) \;\le\; F(a)+F(b)+F(c)$$

(for the given inequality, $F = \sqrt{x}$ is concave).

As to the substitution $a = y+z$, $b = z+x$, $c = x+y$ which was used in the referred to solution and was called the Ravi Substitution, this transformation was known and used before he was born. Geometrically, x, y, z are the lengths which the sides are divided into by the points of tangency of the incircle. Thus, we have the following implications for any triangle inequality or identity:

$$\begin{aligned} F(a,b,c) \;\ge\; 0 \quad &\Longleftrightarrow \quad F(y+z, z+x, x+y) \;\ge\; 0\,,\\ F(x,y,z) \;\ge\; 0 \quad &\Longleftrightarrow \quad F((s-a),(s-b),(s-c)) \;\ge\; 0 \end{aligned}$$

(here s is the semiperimeter). This transformation eliminates the troublesome triangle constraints and lets one use all the machinery for a set of three non-negative numbers.

Another big plus for the Majorization Inequality is that we can obtain both upper and lower bounds subject to other kinds of constraints. Here are two examples:

(1) Consider the bounds on $\sin a_1 + \sin a_2 + \cdots + \sin a_n$ where $n \geq 4$, $\frac{\pi}{2} \geq a_i \geq 0$ and $\sum a_i = S \leq 2\pi$. Since

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, 0, \ldots, 0\right) \succ (a_1, a_2, \ldots, a_n) \succ \left(\frac{S}{n}, \frac{S}{n}, \ldots, \frac{S}{n}\right),$$

we have

$$4 \leq \sin a_1 + \sin a_2 + \cdots + \sin a_n \leq n \sin\left(\frac{S}{n}\right).$$

(2) Consider the bounds on $a_1^2 + a_2^2 + \cdots + a_n^2$ where $\sum a_i = S$ ($\geq n$) and the a_i's are positive integers. Since

$$(S - n + 1, 1, 1, \cdots, 1) \succ (a_1, a_2, \cdots, a_n) \succ \left(\frac{S}{n}, \frac{S}{n}, \cdots, \frac{S}{n}\right),$$

we have

$$(S - n + 1)^2 + n - 1 \geq a_1^2 + a_2^2 + \cdots + a_n^2 \geq n\left(\frac{S}{n}\right)^2.$$

For many other applications, see [1].

Reference

1. A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and Its Applications, Academic Press, NY, 1979.

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Substitutions, Inequalities, and History

Shay Gueron

The Non-Ravi Substitution

A solution to the inequality (1996 Asian Pacific Mathematical Competition)

$$\sqrt{a+b-c}+\sqrt{a-b+c}+\sqrt{-a+b+c} \;\leq\; \sqrt{a}+\sqrt{b}+\sqrt{c} \qquad (1)$$

where a, b, c are the sides of a triangle, appeared in Crux [1999 : 106]. It starts with the substitution $a = x + y$, $b = y + z$, $c = z + x$ (x, y, $z > 0$). This substitution is referred to as the "Ravi Substitution" and reported to be known by this name, at least in Canadian IMO circles.

It seems that this awkward credit for the substitution diffused to wider circles. The same inequality (1) appears in a French problem solving book from 1999 [9, p. 146]. Although the solution proposed in [9] is different, it starts with the same substitution which, amazingly, is called there too the "Ravi Substitution". Further, [9] includes several other mentions and applications of the "Ravi Substitution" [9, pp. 130, 146, 147, 155, 237].

In response to Crux [1999 : 106], Klamkin [6] comments telegraphically that the substitution $a = x+y$, $b = y+z$, $c = z+x$ is a classical technique that was known and used before he (Ravi) was born. It is time to set things straight.

Although Klamkin gives no reference in his piece, he is right; this substitution is one of the first strategies taught in Olympiad training classes, at least — *à la Crux* — in Israeli circles. But since even folklore should be supported by some written evidence, can we find such support? The answer is positive. One example appears in Engel's book (an English translation of previous German versions) [2, p. 178]: the substitution $a = x+y$, $b = y+z$, $c = z + x$ with x, y, $z > 0$ is advice number 7 in Engel's list of 18 possible strategies for proving inequalities. It is explained in [2, p. 164] that this substitution is merely another manifestation of the triangle inequality. The same explanation appears in more detail in [8, Chapter II, p. 26], a book devoted to geometric inequalities. It provides some references to papers where such equivalent forms of the triangle inequality are mentioned (see pp. 35–36). One of these references, namely [5], is a paper from 1971 — written by Klamkin.

Consequently, even without digging into earlier references (which are probably easy to find) Klamkin's remark is evidently correct.

Karamata and the Majorization Inequality

We continue with a historical mood. Klamkin [6] proposes to solve (1) by using the Majorization Inequality. This inequality relates to two sequences $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ and states that: $a_1+\cdots+a_i \le b_1+\cdots+b_i$ for $1 \le i \le n-1$ and $a_1+\cdots+a_n = b_1+\cdots+b_n$ if and only if $f(a_1)+\cdots+f(a_n) \ge f(b_1)+\cdots+f(b_n)$ for any convex function $f(x)$. Clearly, this inequality includes Jensen's Inequality as a special case. The proof of one direction is easy, and the more intricate part can be proved by applying the Abel Summation Formula

$$\sum_{i=1}^{n} a_i b_i = \sum_{j=1}^{n}\left(\sum_{i=1}^{j} a_i\right)(b_j - b_{j+1})\,,$$

setting $b_{n+1} = 0$.

The Majorization Inequality is well known, but unfortunately, this generic name does not reveal its source: this inequality is due to Karamata, 1932 [4], and should therefore be called the Karamata Inequality, as in [1, pp. 31-32]. It turns out to be a strong tool with various applications, some of which can be found in [7] and [8, Chapter VIII]). We also note that before [6], the inequality (1) appeared in [3] (a paper in Hebrew) as an example (probably well known even before) of a case where the Karamata Inequality is a useful approach.

Substitution and the Karamata Inequality

We conclude with an example in the spirit of [6], where a substitution followed by the application of the Karamata Inequality leads to a solution. This is IMO 2000 Problem 2 (as solved by one of the Israeli contestants, Eran Assaf; the problem has some half dozen solutions): Let a, b, c be positive numbers with $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \;\le\; 1\,. \tag{2}$$

The substitution $a = x/y$, $b = y/z$, $c = z/x$ (x, y, $z > 0$) converts (2) to

$$(x - y + z)(y - z + x)(z - x + y) \;\le\; xyz\,. \tag{3}$$

Without loss of generality, assume that $x \le y \le z$, so that $(y + z - x)$, $(z + x - y) > 0$. If $x + y - z \le 0$ (3) follows immediately, so that we may assume that $x + y > z$ as well.

Since $x \ge x + y - z$, $x + y \ge (x + y - z) + (z + x - y)$, and $x + y + z = (x + y - z) + (z + x - y) + (y + z - x)$, we can apply the Karamata Inequality to the triplets (x, y, z) and $(x+y-z, x+z-y, z+y-x)$, and obtain (3) by writing

$$f(x) + f(y) + f(z) \;\le\; f(x + y - z) + f(z + x - y) + f(y + z - x)$$

for the convex function $f(x) = -\ln x$.

Returning to history, it turns out that inequality (3) to which the IMO 2000 problem is reduced/equivalent, is not really a new one: it is due to A. Padoa, in 1925 (Period. Mat. (4)5 : 80–85). Moreover, (3) is equivalent to $a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \leq 3abc$, which is IMO 1964 Problem 2. Could you guess what substitution is helpful for proving these inequalities easily?

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N'oubliez pas d'inclure à toute correspondance votre nom, votre année scolaire, le nom de votre école, ainsi que votre ville, province ou état et pays. Nous sommes surtout intéressés par les solutions d'étudiants du secondaire. Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le *1er septembre 2002*. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

Pour être admissibles au DÉFI MAYHEM de ce mois-ci, les solutions doivent avoir été postées *avant le* 1er juillet 2002, cachet de la poste faisant foi.

34. *Proposé par l'équipe de Mayhem.*

Les nombres **1** à **2002** sont écrits au tableau noir et l'on décide de jouer au jeu suivant :

On lance une pièce de monnaie et on efface deux nombres x et y du tableau. Si l'on tombe sur pile, on écrit $x + y$ au tableau, sinon on écrit $|x - y|$; on continue le processus jusqu'à ce qu'il ne reste plus qu'un nombre. Montrer que ce dernier nombre est impair.

. . . .

The numbers 1 to 2002 are written on a blackboard so you decide to play a fun game. You flip a coin, then erase two numbers, x and y, from the board. If the coin was heads you write the number $x+y$ on the board, if the coin was tails you write the number $|x-y|$. You continue this process until only one number remains. Prove that the last number is odd.

35. *Proposé par l'équipe de Mayhem.*

On définit deux suites par $x_1 = 4732$, $y_1 = 847$, $x_{n+1} = \frac{x_n+y_n}{2}$ and $y_{n+1} = \frac{2x_ny_n}{x_n+y_n}$. Trouver

$$\lim_{n\to\infty} x_n \qquad \text{et} \qquad \lim_{n\to\infty} y_n .$$

. . . .

Two sequences are defined by: $x_1 = 4732$, $y_1 = 847$, $x_{n+1} = \frac{x_n+y_n}{2}$ and $y_{n+1} = \frac{2x_ny_n}{x_n+y_n}$. Find

$$\lim_{n\to\infty} x_n \qquad \text{and} \qquad \lim_{n\to\infty} y_n .$$

36. *Proposé par l'équipe de Mayhem.*

Dans un triangle ABC, soit AM la médiane issue du sommet A. Montrer que $AM \le \frac{AB+AC}{2}$.

. . . .

In $\triangle ABC$, AM is the median from A. Prove $AM \le \frac{AB+AC}{2}$.

37. *Proposé par J. Walter Lynch, Athens, GA, USA.*

Trouver deux entiers positifs différents, plus petits que 100, et tels que la somme des chiffres des deux entiers soit égale au plus grand et que produit de leurs chiffres soit égal au plus petit.

. . . .

Find two (different) positive integers less than 100 such that the sum of the digits in both integers is the larger integer and the product of the digits in both integers is the smaller integer.

38. *Proposé par l'équipe de Mayhem.*

Trouver toutes les valeurs de n telles que $1! + 2! + 3! + \cdots + n!$ soit un carré parfait.

. . . .

Find all values of n such that $1! + 2! + 3! + \cdots + n!$ is a perfect square.

Challenge Board Solutions

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In this issue we present some of the solutions to the Konhauser Problemfest presented in the April 2001 issue [2001 : 204].

1. Last season, the Minnesota Timberwolves won 5 times as many games as they lost, in games in which they scored 100 or more points. On the other hand, in games in which their opponents scored 100 or more points, the Timberwolves lost 50% more games than they won. Given that there were exactly 34 games in which either the Timberwolves or their opponents scored 100 or more points, what was the Timberwolves' win-loss record in games in which *both* they and their opponents scored 100 or more points?

Solution: Let the desired win-loss record be $a - b$, so that of the games when both the Timberwolves and their opponents scored at least 100 points, the Timberwolves won a and their opponents won b. Also, let c be the number of games in which only the Timberwolves scored at least 100 points and d be the number of games in which only their opponents scored at least 100 points. Thus, we have

$$\begin{aligned} a + c &= 5b\,, & (1)\\ b + d &= 1.5a\,, & (2)\\ a + b + c + d &= 34\,. & (3) \end{aligned}$$

Thus, we get $c = 5b - a$, and $d = 1.5a - b$ from (1) and (2), so that (3) yields $3a + 10b = 68$.

Now a and b are non-negative integers, and in particular $b \leq 6$ and $68 - 10b$ is divisible by 3, which is true only for $b = 2$ and $b = 5$. If $b = 2$, then $a = 16$, $c = 5b - 1 = -6 < 0$, which is a contradiction. Thus, we must have $b = 5$ and $a = 6$. The Timberwolves were $6 - 5$ in games in which both they and their opponents scored at least 100 points.

2. Three circles are drawn in chalk on the ground. To begin with, there is a heap of n pebbles inside one of the circles, and there are "empty heaps" (containing no pebbles) in the other two circles. Your goal is to move the entire heap of n pebbles to a different circle, using a series of moves of the following type. For any non-negative integer k, you may move exactly 2^k pebbles from one heap (call it heap **A**) to another (heap **B**), provided that heap **B** begins with fewer than 2^k pebbles, and that after the move, heap **A** ends up with fewer than 2^k pebbles. Naturally, you want to reach your goal in as few moves as possible. For what values of $n \leq 100$ would you need the largest number of moves?

Solution: This is a close relative of the classic "Towers of Hanoi" problem. In that problem, a tapering stack of d discs is to be moved from one peg to one of the other two pegs, which are originally empty. No disc is ever allowed to rest on a larger disc, and the discs must be moved one at a time. If b_d is the number of moves required for d discs, one shows that $b_{d+1} = 2b_d + 1$ (that is, to move the bottom disc, first move the other discs onto a single peg, then after the bottom disc is moved, the others are moved back onto it). However, all we need for our present purpose is that $b_{d+1} > b_d$.

The connection between the given problem and the "Towers of Hanoi" problem can be seen by writing n as a sum of distinct powers of 2. Then, n can be written as such a sum in exactly one way.

The distinct powers of 2 correspond to the discs in the "Towers of Hanoi" problem, but *larger powers of* 2 *correspond to smaller discs*. The condition for moving 2^k pebbles corresponds to the fact that the top disc must be moved before any of the discs underneath.

Note that d corresponds to the number of distinct powers of 2 that add up to n (or the number of 1's in the binary expansion of n). So our problem boils down to: what numbers $n \leq 100$ have the most digits 1 in their binary expansion?

The powers of 2 that can be used in the binary expansion of numbers $n \leq 100$ are 2^6, 2^5, 2^4, 2^3, 2^2, 2^1, 2^0. However, they cannot *all* be used since their sum is $127 > 100$. It is possible to use all but one of them in just two ways:

$$\begin{aligned} 2^6 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 &= 95\,, \\ 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 &= 63\,. \end{aligned}$$

Hence, 63 *and* 95 are the desired values of n.

3. (a) Begin with a string of 10 A's, B's, and C's, for example

```
A B C C B A B C B A
```

and underneath, form a new row, of length 1 shorter, as follows: between two consecutive letters that are different, you write the third letter, and between two letters that are the same, you write that same letter again. Repeat this process until you have only one letter in the new row. For example, for the string above, you will now have:

```
A B C C B A B C B A
 C A C A C C A A C
  B B B B C B A B
   B B B A A C C
    B B C A B C
     B A B C A
      C C A B
       C B C
        A A
         A
```

Prove that the letters at the corners of the resulting triangle are always either all the same or all different.

Solution: Think of the letters A, B, C as representing the numbers 0, 1, 2, respectively, $\pmod 3$. Then, if in some row we have $\cdots\ x\ \ y\ \cdots$ where x and y are integers $\pmod 3$, we get $\cdots -x-y \cdots$ in the next row, where $-x-y$ is computed $\pmod 3$. If the original row corresponds to the integers $x_1\ \ x_2\ \ \cdots\ \ x_{10}\ \pmod 3$, then the next rows are:

$$\begin{array}{ccccccc} -x_1-x_2 & & -x_2-x_3 & & -x_3-x_4 & & \cdots \\ & x_1+2x_2+x_3 & & x_2+2x_3+x_4 & & \cdots & \\ & & -x_1-3x_2-3x_3-x_4 & & \cdots & & \end{array}$$

and we see the pattern (which can be proved by induction): apart from the $\pm$ signs, the coefficients are those of Pascal's triangle, the binomial coefficients. In particular, the tenth (bottom) row will consist of the single entry

$$-x_1-\binom{9}{1}x_2-\binom{9}{2}x_3-\cdots-\binom{9}{8}x_9-x_{10} \pmod 3 .$$

But since 9 is a power of the prime 3, all binomial coefficients $\binom{9}{1}$, $\binom{9}{2}$, $\ldots$, $\binom{9}{8}$ are divisible by 3, so that the entry in the bottom row is equal to $-x_1-x_{10} \pmod 3$. Thus, the three corner numbers $(x_1, x_{10}, -x_1-x_{10}) \pmod 3$ that add to $0 \pmod 3$ and it follows that they are either all the same or all different.

(b) For which positive integers n (besides 10) is the result from part (a) true for all strings of n A's, B's, and C's?

Solution: For $n=1$, and for $n=3^k+1$, $k \geq 0$. For such an n, the n^{th} row will have the form:

$$-x_1-\binom{3^k}{1}x_2-\binom{3^k}{2}x_3-\cdots-\binom{3^k}{3^k-1}x_{n-1}-x_n \equiv -x_1-x_n \pmod 3 ,$$

because the binomial coefficients are divisible by 3.

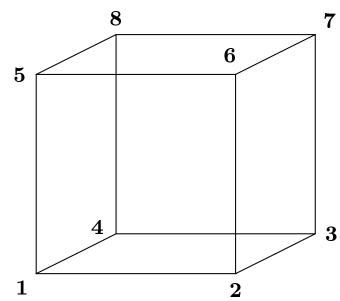
4. When Mark climbs a staircase, he ascends either 1, 2, or 3 stairsteps with each stride, but in no particular pattern from one foot to the next. In how many ways can Mark climb a staircase of 10 steps? (Note that he must finish on the top step. Two ways are considered the same if the number of steps for each stride are the same; that is, it does not matter whether he puts his best or his worst foot forward first.) Suppose that a spill has occurred on the 6^{th} step and Mark wants to avoid it. In how many ways can he climb the staircase without stepping on the 6^{th} step?

Solution: Let a_n be the number of ways for Mark to climb a staircase of n steps. Then $a_1=1$, $a_2=2$ and $a_3=4$. For $n>3$, consider Mark's last stride. If his last stride was 1–step, then before that he climbed an $n-1$–step staircase, and there are a_{n-1} ways in which that can be done. Using a similar argument for 2– and 3–step last steps, we get $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n > 3$. Using this we can calculate $a_{10} = 274$. Thus there are 274 ways that Mark can climb the stairs before the spill has occurred.

Once the spill has occurred, we can work through the same way. If b_n is the number of ways to get to step n and not step on step 6, then we have: $b_n = a_n$ for $n \leq 5$, $b_6 = 0$. Thus, using the recurrence relation for a_n we get $b_7 = 0 + 13 + 7 = 20$, $b_8 = 20 + 0 + 13 = 33$, $b_9 = 0 + 20 + 33 = 53$ and $b_{10} = 20 + 33 + 53 = 106$. Thus, there are 106 ways to go up the stairs and avoid the spill.

5. Number the vertices of a cube from 1 to 8. Let A be the 8×8 matrix whose (i, j) entry is 1 if the cube has an edge between vertices i and j, and is 0 otherwise. Find the eigenvalues of A, and describe the corresponding eigenspaces.

Solution:Let the vertices be numbered as shown.



Then we have

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

We can find a few eigenvectors by inspection; for example $(1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)^T$ is an eigenvector for $\lambda = 3$ and $(1\ \ 1\ \ 1\ \ 1\ \ -1\ \ -1\ \ -1\ \ -1)^T$ is an eigenvector for $\lambda = -1$. From these a pattern starts to emerge. Note that the entries 1 occur in positions 1, 2, 3, 4 and the entries -1 occur in positions 5, 6, 7, 8. Now, $\{1,\ 2,\ 3,\ 4\} \cup \{5,\ 6,\ 7,\ 8\} = S_1 \cup S_2$ is a decomposition of the vertex set $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\}$ into two disjoint parts S_1, S_2 with the property that each vertex in S_i is connected to two vertices in S_i and one vertex in S_j for $i,\ j \in \{1, 2\}$, $i \neq j$. We now have the observation, whose proof follows from the definition of A given in the problem:

Let $S = S_1 \cup S_2$ be any decomposition of $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\}$ into disjoint subsets S_1, S_2 such that there exist integers k and l with $k + l = 3$ such that each vertex in S_i is connected to k vertices in S_i and l vertices in S_j for $i,\ j \in \{1,\ 2\}$, $i \neq j$. Then the vector with entries 1 in the positions corresponding to S_1 and -1 in the positions corresponding to S_2 is an eigenvector of A for the eigenvalue $k - l$.

Note that while we could interchange the subsets S_1 and S_2, that would just change the eigenvector to its opposite. Except for this, there are eight different decompositions of S of the desired type, as shown in the table below.

S_1	S_2	k	l	λ	Eigenvector
S	ϕ	3	0	3	$(1,1,1,1,1,1,1,1)^T$
$\{1,\ 2,\ 3,\ 4\}$	$\{5,\ 6,\ 7,\ 8\}$	2	1	1	$(1,1,1,1,-1,-1,-1,-1)^T$
$\{1,\ 2,\ 5,\ 6\}$	$\{3,\ 4,\ 7,\ 8\}$	2	1	1	$(1,1,-1,-1,1,1,-1,-1)^T$
$\{1,\ 4,\ 5,\ 8\}$	$\{2,\ 3,\ 6,\ 7\}$	2	1	1	$(1,-1,-1,1,1,-1,-1,1)^T$
$\{1,\ 2,\ 7,\ 8\}$	$\{3,\ 4,\ 5,\ 6\}$	1	2	-1	$(1,1,-1,-1,-1,-1,1,1)^T$
$\{1,\ 4,\ 6,\ 7\}$	$\{2,\ 3,\ 5,\ 8\}$	1	2	-1	$(1,-1,-1,1,-1,1,1,-1)^T$
$\{1,\ 3,\ 5,\ 7\}$	$\{2,\ 4,\ 6,\ 8\}$	1	2	-1	$(1,-1,1,-1,1,-1,1,-1)^T$
$\{1,\ 3,\ 6,\ 8\}$	$\{2,\ 4,\ 5,\ 7\}$	0	3	-3	$(1,-1,1,-1,-1,1,-1,1)^T$

(Geometrically, the eigenvectors for $\lambda = 1$ in this table can be thought of as corresponding to pairs of opposite faces of the cube; the eigenvectors for $\lambda = -1$ can be thought of as corresponding to pairs of diagonal planes through the cube.) It is easily checked that the eigenvectors for $\lambda = 1$ (similarly for $\lambda = -1$) listed above are linearly independent. Therefore, for matrix A we have eigenvalues $\lambda = 3$, $\lambda = 1$ (with multiplicity 3), $\lambda = -1$ (with multiplicity 3) and $\lambda = -3$, and the eigenspaces are spanned by the vectors in the table.

6. Let $f(x)$ be a twice-differentiable function on the open interval $(0, 1)$ such that

$$\lim_{x \to 0^+} f(x) = -\infty\,, \quad \text{and} \quad \lim_{x \to 1^-} f(x) = +\infty\,.$$

Show that $f''(x)$ takes on both negative and positive values.

Solution: Take an arbitrary number x_0 in the interval $(0, 1)$ and let $f'(x_0) = m$. Suppose that $f''(x) \geq 0$ for all x in $(0, 1)$. Then, in particular, for all x with $0 < x < x_0$ we must have $f(x) \geq f(x_0) + m(x - x_0)$.

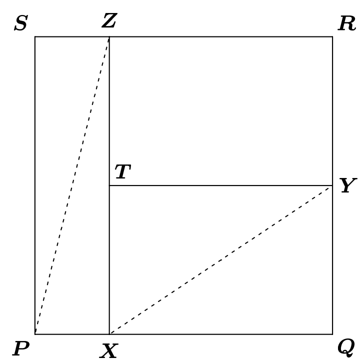
But $f(x) \geq f(x_0) + m(x - x_0)$ shows that, as $x \to 0^+$, we have $f(x) \geq f(x_0) - mx_0$, contradicting the given $\lim_{x \to 0^+} f(x) = -\infty$.

Similarly, if $f''(x) \leq 0$ for all x in $(0, 1)$ then for all x with $x_0 < x < 1$ we must have $f(x) \leq f(x_0) + m(x - x_0)$, and this contradicts $\lim_{x \to 1^-} f(x) = +\infty$.

Thus, we must have $f''(x) > 0$ for some x in $(0, 1)$ as well as $f''(x) < 0$ for some x in $(0, 1)$, and we are done.

7. Three stationary sentries are guarding an important public square which is, in fact, square, with each side measuring 8 rods (recall that one rod equals 5.5 yards). (If any of the sentries see trouble brewing at any location on the square, the sentry closest to the trouble spot will immediately cease to be stationary and dispatch to that location. And like all good sentries, these three are continually looking in all directions for trouble to occur.) Find the maximum value of D, so that no matter how the sentries are placed, there is always some spot in the square that is at least D rods from the closest sentry.

Solution: Divide the square into rectangles as shown, in such a way that $QY = YR = 4$ and $PZ = XY$. We claim once this has been done $D_0 = \frac{1}{2}XY$ is the maximum value of D.



Proof: One way to place the sentries is at the centres of the rectangles $PXZS$, $XQYT$, $TYRZ$; for this placement, every point on the square is at most D_0 from the sentry at the centre of the rectangle it belongs to. Thus, a value $D > D_0$ is not possible. To show that $D = D_0$ actually works, we have to show that there is no placement of the sentries for which every point of the square has distance less than D_0 to the closest sentry. Suppose we did have such a placement. Since there are four corners P, Q, R, S of the square and only three sentries, at least two corners would be guarded by the same sentry. Without loss of generality, let P and S be two corners guarded by the same sentry. Now note that X and Y cannot be guarded by a single sentry, because by our assumption that sentry would have distance less than D_0 to both X and Y, so that $XY < 2D_0 = XY$, a contradiction. Similarly, no other two opposite corners of the three smaller rectangles that we constructed can be guarded by a single sentry. Thus, S, X, Y are all guarded by different sentries, say 1, 2, 3, in that order, and since S and P are guarded by the same sentry P, Z, Y are also guarded by 1, 2, 3 in that order. Thus, Q and R must be guarded by 3. But, we will show that the mid-point O of XQ is too far from Z and R to be guarded by 2 and 3, respectively, which will provide a contradiction.

Let $PX = a$, $XQ = 8 - a$. Then

$$PZ = XY \quad \Longleftrightarrow \quad PX^2 + XZ^2 = XQ^2 + QY^2 ,$$

which gives $a = 1$, so that $PX = 1$, $QX = 7$, $XO = \frac{7}{2}$ and $ZO > ZP = 2D_0$, showing that O is indeed too far from Z. Thus, the answer is $D_0 = \frac{1}{2}\sqrt{65}$.

8. The Union Atlantic Railway is planning a massive project: a railroad track joining Cambridge, Massachusetts and Northfield, Minnesota. However, the funding for the project comes from the will of Orson Randolph Kane, the eccentric founder of the U.A.R., who has specified some strange conditions on the railway; thus the skeptical builders are unsure whether or not it is possible to build a railway subject to his unusual requirements.

Kane's will insists that there must be exactly 100 stops (each named after one of his great-grandchildren) between the termini, and he even dictates precisely what the distance along the track between each of these stops must be. (Unfortunately, the tables in the will *do not* list the order in which the stops are to appear along the railway.) Luckily, it is clear that Kane has put some thought into these distances; for any three distinct stops, the largest of the three distances among them is equal to the sum of the smaller two, which is an obvious necessary condition for the railway to be possible. (Also, all the given distances are shorter than the distance along a practical route from Cambridge to Northfield!)

U.A.R.'s engineers have pored over the numbers and noticed that for any four of Kane's stops, it would be possible to build a railway with these four stops and the distances between them as Kane specifies. Prove that, in fact, it is possible to complete the entire project to Kane's specifications.

Solution: Let $d(i, j)$ be the required distance from stop i to stop j along the track. Note that if four stops are arranged successfully along the track, there is a unique greatest distance between two of those four stops: the one that ends up closest to Cambridge, and the one that ends up closest to Northfield.

Now for the proof.

The case $n = 4$ is trivial. Suppose the statement is true for N stops ($N \geq 4$) and we have a table of distances $d(i, j)$ for $N + 1$ stops satisfying the given conditions. Find the largest of these distances. This largest distance can only occur *once*, because if $d(i, j) = d(k, l)$ with $\{i, j\} \neq \{k, l\}$, it would be impossible to arrange stops i, j, k, l (or if two of them were equal, say $j = l$, then stops i, j, k, m with $m \neq i, j, k$) successfully along the track. Without loss of generality, assume that the greatest distance in the table is $d(1, N + 1)$. Let a, b be any two distinct stops with $2 \leq a, b \leq N$. Then, since stops 1, a, b, $N+1$ could be arranged successfully, and $d(1, N+1)$ is the largest distance, we must have either $d(1,a) = d(1,b) + d(a,b)$ or $d(1,b) = d(1,a) + d(a,b)$ (but we cannot have $d(a,b) = d(1,a) + d(1,b)$). Now, ignore stop $N+1$ for the moment and arrange stops 1, 2, $\ldots$, N successfully along the track, which is possible by the induction hypothesis. Since we cannot have $d(a,b) = d(1,a) + d(1,b)$ for any a, $b \in \{2, 3, \ldots, N\}$, we have that 1 cannot be between any two other stops in this successful arrangement; that is, all stops 2, 3, $\ldots$, N must be the same side of stop 1. Since they all have the proper distances to stop 1 and $d(1, N+1)$ is larger than all those distances, we can place stop $N+1$ on the track in such a way that all stops 2, 3, $\ldots$, N are between stop 1 and stop $N+1$, that stops 1 and $N+1$ are the proper distance apart, and that all stops 1, 2, $\ldots$, N are still in the same places.

All we now need to do is to check that stop $N+1$ is the correct distance $d(a, N+1)$ to each of the stops a, $2 \le a \le N$. Then, since $d(1, N+1)$ is less than the distance from Cambridge to Northfield, we can arrange for the stops to follow each other in this order between Cambridge and Northfield, and we will be done.

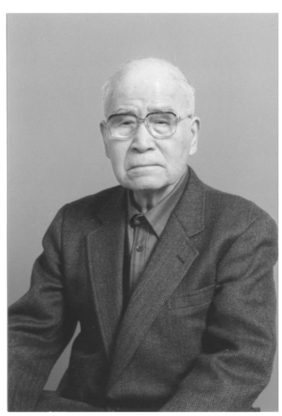
Consider any stop a, $2 \le a \le N$. Since it has the correct distance $d(a,1)$ to stop 1, in the above placement its distance to stop $N+1$ is given by $d(1, N+1) - d(a,1)$. But since $d(1, N+1)$ is the largest of the required distances between stops 1, a, $N+1$, we have

$$d(1, N+1) \;=\; d(a,1) + d(a, N+1)\,,$$

so that

$$d(1, N+1) - d(a,1) \;=\; d(a, N+1)\,,$$

and we are done.



Professor Toshio Seimiya

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. (a) Given positive numbers a_1, a_2, a_3, $\cdots$, a_n and the quadratic function $f(x) = \sum_{i=1}^{n} (x - a_i)^2$, show that $f(x)$ attains its minimum value at $\frac{1}{n}\sum_{i=1}^{n} a_i$, and prove that $\sum_{i=1}^{n} a_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2$.

(b) The sum of sixteen positive numbers is 100 and the sum of their squares is 1000. Prove that none of the sixteen numbers is greater than 25.

(1996 Canadian Open, Problem B3)

Solution. (a) The quadratic function $f(x)$ is a parabola, and the graph $y = f(x)$ opens upward. (The x^2–coefficient is positive.) Hence the vertex of the graph is a minimum point; that is, there is a unique value of x that minimizes $y = f(x)$, and the point (x, y) is the vertex.

It is known that the x–coordinate of the vertex of the function $f(x) = ax^2 + bx + c$ is $-b/2a$. Our given function, after expanding, is $f(x) = nx^2 - 2x\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} a_i^2$, and the x-coordinate of the vertex is $\frac{1}{n}\sum_{i=1}^{n} a_i$. Hence, for this value of x, the value of $f(x)$ is minimized.

We also know that $f(x)$ is greater than or equal to 0, since it is the sum of non-negative squares. Hence, the discriminant must be less than or equal to 0. (This condition corresponds to $f(x)$ having one or no roots.)

If we let the discriminant be D, then $D/4 = \left(\sum_{i=1}^{n} a_i\right)^2 - n\cdot\sum_{i=1}^{n} a_i^2 \leq 0$.
Rearranging this inequality gives us the desired result, $\sum_{i=1}^{n} a_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2$.

(b) Let the largest value of the a_i's be b. Consider the 15 a_i's, excluding b. Then, apply the result in (a) to these 15 numbers. We have $\sum a_i^2 - \frac{1}{15}\left(\sum a_i\right)^2 \geq 0$, where both summations are taken over the 15 a_i's excluding b.

$$\begin{aligned}\sum a_i^2 - \frac{1}{15}\left(\sum a_i\right)^2 &= \frac{1}{15}\cdot\left\{15\left(1000 - b^2\right) - (100 - b)^2\right\} \\ = \frac{1}{15}\cdot\left(-16b^2 + 200b + 5000\right) &= -\frac{8}{15}\cdot(b - 25)(2b + 25) \geq 0\,.\end{aligned}$$

Since b is a positive value, the inequality holds true if and only if $b \leq 25$. In other words, the largest of the a_i's must not exceed 25, QED.

SKOLIAD No. 60

Shawn Godin

Solutions may be sent to Shawn Godin, Cairine Wilson S.S., 975 Orleans Blvd., Orleans, ON, CANADA, K1C 2Z5, or emailed to

mayhem-editors@cms.math.ca

Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 July 2002*. A copy of ***MATHEMATICAL MAYHEM*** **Vol. 2** will be presented to the pre-university reader(s) who send in the best set of solutions before the deadline. The decision of the editor is final.

This issue's item comes to us from Manitoba. My thanks go to Diane Dowling at St.Paul's College in Winnipeg for forwarding the material to me.

MANITOBA MATHEMATICAL CONTEST, 2001

For Students in Senior 4

9:00 a.m. – 11:00 a.m. Wednesday, February 21, 2001

Sponsored by

The Actuaries' Club of Winnipeg, The Manitoba Association of Mathematics Teachers, The Canadian Mathematical Society, and The University of Manitoba

Answer as much as possible. You are not expected to complete the paper. See both sides of this sheet. Hand calculators are not permitted. Numerical answers only, without explanation, will not be given full credit.

1. (a) Solve the equation $\frac{6}{x} - \frac{1}{x^2} = 9$.

(b) Find real numbers a and b such that $(2^a)(4^b) = 8$ and $a+7b = 4$.

2. (a) Find all real numbers x such that $25|x| = x^2 + 144$.

(b) If x is a real number and $\frac{5}{\sqrt{7+3\sqrt{x}}} = \sqrt{7-3\sqrt{x}}$ what is the value of x?

3. (a) If two of the roots of $x^3+px^2+qx+r = 0$ are equal in absolute value but opposite in sign, prove that $pq = r$. (p, q and r are real numbers.)

(b) If a, b and c are real numbers such that $a + b + c = 14$, $c^2 = a^2 + b^2$ and $ab = 14$, find the numerical value of c.

4. (a) If $0^\circ < \theta < 180^\circ$ and $2\sin^2\theta + 3\sin\theta \geq 2$ what is the largest possible value of θ?

(b) Let O be the origin, P the point whose coordinates are $(2, 3)$ and F a point on the line whose equation is $y = \frac{x}{2}$. If PF is perpendicular to OF find the coordinates of F.

5. How many consecutive zeros are there at the end of the product of all the integers from 16 to 100 inclusive?

6. In triangle ABC, $\angle ACB = 135^\circ$, $CA = 6$ and $BC = \sqrt{2}$. If M is the mid-point of the side AB, find the length of CM.

7. A circle of radius 2 has its centre in the first quadrant and has both coordinate axes as tangents. Another smaller circle also has both coordinate axes as tangents and has exactly one point in common with the larger circle. Find the radius of the smaller circle.

8. A parallelogram has an area of 36 and diagonals whose lengths are 10 and 12. Find the lengths of its sides.

9. a, b, c, d are distinct integers such that $(x-a)(x-b)(x-c)(x-d)=4$ has an integral root r. Prove that $a + b + c + d = 4r$.

10. If x, y and z are positive real numbers, prove that

$$(x + y - z)(x - y)^2 + z(x - z)(y - z) \ \geq \ 0 \,.$$

Next we present the official solutions to the Mandelbrot competitions from the October issue [2001 : 385].

The Mandelbrot Competition
Division B Round Two Individual Test
December 1997

1. If a group of positive integers has a sum of 8, what is the greatest product the group can have? **(1 point)**

Solution.

Clearly we do not want any 1's in our group, since they contribute to the sum but not the product. Any 5 in the group can be replaced by a 2 and a 3, which have the same sum but a greater product. Similarly, any 6 can be replaced by two 3's, and so on, since every number contributes more to the product if broken down into 2's and 3's. (A 4 can be replaced with two 2's, with no effect on the sum or product.) Hence, our optimal group should consist of 2's and 3's only. The only such sets of numbers summing to 8 are $2+2+2+2$ or $2+3+3$. The products in these cases are 16 and 18. Hence, $\boxed{18}$ is optimal.

2. There is one two-digit number such that if we add 1 to the number and reverse the digits of the result, we obtain a divisor of the number. What is the number? **(1 point)**

Solution.

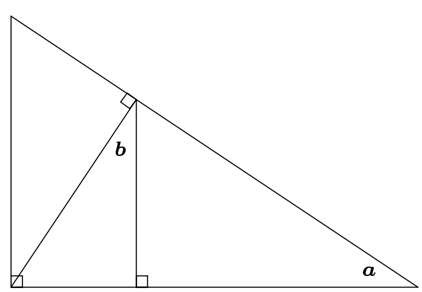
Let the two-digit number n be $\boxed{\boxed{t|u}}$. If u is not 9, then the number obtained by adding one and reversing the digits is $\boxed{\boxed{u+1|t}}$. Since this potential divisor cannot equal n, it must be half of n or less, so that $2(u+1) \leq t$ and $2\left((u+1)\times 10+t\right) \leq t \times 10 + u$. This restricts us to the numbers $n = 30$, 40, 50, 51, 60, 61, 70, 71, 80, 81, 82, 90, 91, or 92. A quick check shows that none of these numbers works. Hence, u must be 9. If $n = \boxed{\boxed{t|9}}$, then $n+1 = \boxed{\boxed{t+1|0}}$ (unless $t = 9$), so that the divisor would be $t+1$. But n equals $10(t+1)-1$, which cannot be a multiple of $t+1$. Thus t must also be 9, yielding $n = \boxed{99}$, making the divisor equal to 001, or 1.

3. Ten slips of paper, numbered 1 through 10, are placed in a hat. Three numbers are drawn out, one after another. What is the probability that the three numbers are drawn in increasing order? **(2 points)**

Solution.

Let the numbers chosen be A, B, and C. There are six orders in which the slips can be chosen: ABC, ACB, BAC, BCA, CAB, CBA. Of these six, only one is in the increasing order we desire. Hence, the probability is $\boxed{\frac{1}{6}}$.

4. The three marked angles are right angles. If $\angle a = 20°$, then what is $\angle b$? **(2 points)**



Solution.

Note that $\angle b$ is complementary to an angle which is complementary to $\angle a$; hence, $\angle b = \angle a$, so that $\angle b = \boxed{20°}$.

5. Vicky asks Charlene to identify all non-congruent triangles $\triangle ABC$ given:

(a) the value of $\angle A$

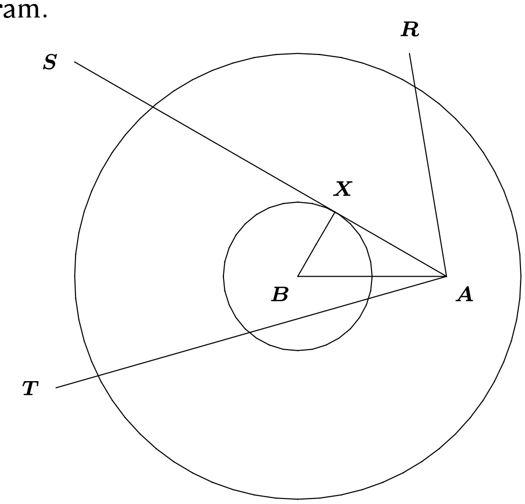
(b) $AB = 10$, and

(c) length BC equals either 5 or 15.

Charlene responds that there are only two triangles meeting the given conditions. What is the value of $\angle A$? **(2 points)**

Solution.

Consider $\overline{AB}$ to be a fixed segment of length 10. Since BC is either 5 or 15, C must lie on one of the two circles with center B and radii 5 and 15, as in the diagram.



There are many possible values for $\angle A$; the possible positions for point C are the points where $\angle A$ intersects the two circles. For some angles, as with $\angle A = \angle BAR$ in the diagram, there is only one possible point C; for others, as with $\angle A = \angle BAT$, there are three. The only value which gives exactly two points C is that which makes the angle tangent to the inner circle, as with $\angle A = \angle BAS$. Call the point of tangency X. Since $AB = 10$, $BX = 5$, and $\angle BXA$ is a right angle, we conclude that triangle ABX is a $30°$-$60°$-$90°$ triangle. Hence, $\angle A = \boxed{30°}$.

6. Five pirates find a cache of five gold coins. They decide that the shortest pirate will become bursar and distribute the coins — if half or more of the pirates (including the bursar) agree to the distribution, it will be accepted; otherwise, the bursar will walk the plank and the next shortest pirate will become bursar. This process will continue until a distribution of coins is agreed upon. If each pirate always acts so as to stay aboard if possible and maximize his wealth, and would rather see another pirate walk the plank than not (all else being equal), then how many coins will the shortest pirate keep for himself? **(3 points)**

Solution.

Call the pirates $p_1, p_2, \ldots, p_5$, with p_1 the shortest and p_5 the tallest. Consider what would happen if only p_4 and p_5 remained. Whatever division strategy p_4 suggested would hold, since p_4's vote alone would constitute half the total vote. Thus, p_4 would simply allot himself all the gold. Next consider the situation where three pirates remained. Whatever distribution p_3 chose, p_5 would have to agree, as long as he got one or more coins. His only alternative would be to go to the two-pirate situation, in which p_5 gets nothing at all. Hence, p_3 would simply take 4 coins for himself and allot 1 coin to p_5, getting a majority vote from himself and p_5. Similarly, with four pirates remaining, p_2, the bursar, would take 4 coins for himself and allot 1 coin to p_4. Again, since p_4 would have otherwise gotten nothing, he would have to support the plan. With five pirates, the bursar, p_1, would allot himself $\mathbf{3}$ coins and give one coin each to p_3 and p_5. Since each of p_3 and p_5 gets more than they would get by vetoing the plan, they must support it. The shortest pirate gets $\boxed{3}$ coins.

7. The twelve positive integers $a_1 \leq a_2 \leq \cdots \leq a_{12}$ have the property that no three of them can be the side lengths of a non-degenerate triangle. Find the smallest possible value of $\frac{a_{12}}{a_1}$. **(3 points)**

Solution.

If a, b and c can be sides of a non-degenerate triangle with $a \leq b \leq c$, we always must have $c < a + b$. Hence, if no three of our integers can form a non-degenerate triangle, we must have $a_i \geq a_j + a_k$ for any three with $i > j, k$. Since the numbers are increasing, it suffices to show that

$$a_1 + a_2 \ \leq \ a_3\,, \quad a_2 + a_3 \ \leq \ a_4\,, \quad a_3 + a_4 \ \leq \ a_5$$

and so on. Substituting the first inequality into the second, we have

$$a_1 + 2a_2 \ \leq \ a_4\,.$$

Substituting this and the second inequality into the third, we get

$$2a_1 + 3a_2 \ \leq \ a_5\,.$$

Substituting into the fifth inequality:

$$3a_1 + 5a_2 \ \leq \ a_6\,.$$

We continue in this manner. At each step, we simply increase the coefficients of a_1 and a_2; they must increase as Fibonacci numbers, since at each step we get the new coefficient by adding the previous two. Hence, the final inequality will read

$$55a_1 + 89a_2 \ \leq \ a_{12}\,.$$

To obtain the smallest possible ratio we choose $a_1 = a_2$. This simply yields $144a_1 \leq a_{12}$, or $a_{12}/a_1 \geq 144$. Since this limit can be attained, with $a_1 = a_2 = 1$, $a_3 = 2$, $a_4 = 3$, $a_5 = 5$, $\ldots$, $a_{12} = 144$, we find that $\boxed{144}$ is the least value.

The Mandelbrot Competition
Division B Round Two Team Test
December 1997

FACTS: A polynomial $p(x)$ of degree n or less is determined by its value at $n+1$ x–coordinates. For $n = 1$ this is a familiar statement; a line (degree one polynomial) is determined by two points. Moreover, the value of $p(x)$ at any other x–value can be computed in a particularly nice way using Lagrange interpolation, as outlined in the essay *An Interpretation of Interpolation*.

We will also need a result from linear algebra which states that a system of n "different" linear equations in n variables has exactly one solution. For example, there is only one choice for x, y, and z which satisfies the equations $x+y+z=1$, $x+2y+3z=4$, and $x+4y+9z=16$.

SETUP: Let $p(x)$ be a degree three polynomial for which we know the values of $p(1)$, $p(2)$, $p(4)$, and $p(8)$. By the facts section there is exactly one such polynomial. According to Lagrange interpolation the number $p(16)$ can be deduced; it equals

$$p(16) \;=\; A_0 p(1) + A_1 p(2) + A_2 p(4) + A_3 p(8)\,,$$

for some constants A_0 through A_3. The goal of this team test will be to compute the A_i and use them to find information about $p(16)$ *without ever finding an explicit formula for* $p(x)$.

Problems:

Part i: (4 points) We claim that the A_i can be found by subtracting

$$x^4 - (x-1)(x-2)(x-4)(x-8) \;=\; A_3x^3 + A_2x^2 + A_1x + A_0\,. \tag{1}$$

Implement this claim to compute A_0 through A_3.

Solution.

Apparently we have been handed a magic formula which generates the constants A_i needed for computing $p(16)$. It is referred to frequently in this solution, so we reproduce it here:

$$x^4 - (x-1)(x-2)(x-4)(x-8) = A_3x^3 + A_2x^2 + A_1x + A_0\,. \tag{2}$$

Rather than marvel at this stroke of good fortune so early on, we set about computing the values A_0 through A_3. Carefully multiplying out the left-hand side yields

$$\begin{aligned} &x^4 - (x-1)(x-2)(x-4)(x-8) \\ &\quad= \; x^4 - (x^2-3x+2)(x^2-12x+32) \\ &\quad= \; x^4 - (x^4-15x^3+70x^2-120x+64) \\ &\quad= \; 15x^3 - 70x^2 + 120x - 64 \\ &\quad= \; A_3x^3 + A_2x^2 + A_1x + A_0\,. \end{aligned}$$

Therefore we should use $A_3 = 15$, $A_2 = -70$, $A_1 = 120$, and $A_0 = -64$.

Part ii: (4 points) To show that these A_i are in fact the correct numbers we must show that they correctly predict $p(16)$ for four "different" polynomials. We begin with the case $p(x) = x$. Show that the value of $p(16)$ agrees with the prediction $A_0p(1) + A_1p(2) + A_2p(4) + A_3p(8)$. (HINT: try $x = 2$ in (1).)

Solution.

According to Lagrange Interpolation, if $p(x)$ is a polynomial of degree three or less, then we should be able to predict $p(16)$ based on the values of $p(x)$ at $x = 1$, 2, 4 and 8. We will now show that the constants A_i just computed do the job by showing that $p(16)$ always equals the sum $A_3p(8) + A_2p(4) + A_1p(2) + A_0p(1)$. According to the facts section we need only verify that the A_i work in four different cases to know that they will always work. In checking these four cases we employ a somewhat clever method that never actually uses the numbers calculated in part i, just the equation (2) that produced them.

First suppose that $p(x) = x$. Then clearly $p(1) = 1$, $p(2) = 2$, $p(4) = 4$, and $p(8) = 8$. Let us check whether or not $A_3p(8) + A_2p(4) + A_1p(2) + A_0p(1)$, which is the same as $8A_3 + 4A_2 + 2A_1 + A_0$, correctly predicts $p(16)$. Substituting $x = 2$ into equation (2) gives us

$$2^4 - (2-1)(2-2)(2-4)(2-8) \;=\; 8A_3 + 4A_2 + 2A_1 + A_0\,.$$

The left-hand side equals 16 since the $(2-2)$ factor causes the second term to vanish. However, the right hand side is our prediction for $p(16)$. Sure enough, we get $p(16) = 16$, just as we should for the function $p(x) = x$.

Part iii: (5 points) Continuing the previous part, show that the A_i correctly predict $p(16)$ for the three other polynomials $p(x) = 1$, $p(x) = x^2$ and $p(x) = x^3$.

Solution.

In all cases the prediction for $p(16)$ is $A_3p(8) + A_2p(4) + A_1p(2) + A_0p(1)$. Continuing our work from part ii we try $p(x) = 1$, so our prediction becomes $A_3 + A_2 + A_1 + A_0$. Using $x = 1$ in (2) yields

$$1^4 - (1-1)(1-2)(1-4)(1-8) \;=\; A_3 + A_2 + A_1 + A_0\,.$$

The left-hand side reduces to 1, so that the prediction is $p(16) = 1$, which again is correct. When $p(x) = x^2$ the prediction for $p(16)$ becomes $64A_3 + 16A_2 + 4A_1 + A_0$. This can be found quickly by substituting $x = 4$ into equation (2):

$$4^4 - (4-1)(4-2)(4-4)(4-8) \;=\; 64A_3 + 16A_2 + 4A_1 + A_0\,.$$

The now familiar cancellation occurs on the left hand side, leaving us with a prediction of 4^4 for $p(16)$. Since $p(x) = x^2$ we expect to have $p(16) = 16^2$, and indeed $16^2 = (4^2)^2 = 4^4$. The case of $p(x) = x^3$ works in exactly the same manner, substituting $x = 8$ into equation (2), so we encourage the reader to try it as practice. (Naturally teams were expected to show the details for this case as well in their solutions!)

Notice that we were able to do all of our checking without ever using the numerical values of A_0 through A_3. The other more obvious method is to plug in the values for the A_i and do the arithmetic. However, the slick technique can be generalized, while the more routine method cannot.

Part iv: (4 points) Suppose that $p(x)$ is a third degree polynomial with $p(1) = 0$, $p(2) = 1$, and $p(4) = 3$. What value should $p(8)$ have to guarantee that $p(x)$ has a root at $x = 16$?

Solution.

We have now verified that $A_3 = 15$, $A_2 = -70$, $A_1 = 120$, and $A_0 = -64$ are the correct values needed to interpolate $p(16)$. We are also told in this problem that $p(1) = 0$, $p(2) = 1$, and $p(4) = 3$. Furthermore, we want $p(16) = 0$ so that $p(x)$ has a root at $x = 16$. Substituting all of these values into our interpolation formula produces

$$\begin{array}{rrcl} & p(16) & = & A_3p(8) + A_2p(4) + A_1p(2) + A_0p(1) \\ \Longrightarrow & 0 & = & 15p(8) - 70 \cdot 3 + 120 \cdot 1 - 64 \cdot 0 \\ \Longrightarrow & 90 & = & 15p(8) \\ \Longrightarrow & 6 & = & p(8)\,. \end{array}$$

Thus we need $p(8) = 6$ to ensure that $p(16) = 0$ so that there is a root at $x = 16$.

Part v: (4 points) Let $p(x)$ be a degree three polynomial with $p(1) = 1$, $p(2) = 3$, $p(4) = 9$, and $p(8) = 27$. Calculate $p(16)$. How close does it come to the natural guess of 81?

Solution.

There are two ways to do this problem—a long way and a short way. The long way involves substituting all of the values for A_0 through A_3 and $p(1)$ through $p(8)$ into the interpolation formula shown above and cranking out the answer. The short way, hinted at by our above work, involves plugging $x = 3$ into equation (2). We opt for the short way, obtaining

$$3^4 - (3-1)(3-2)(3-4)(3-8) \;=\; 27A_3 + 9A_2 + 3A_1 + A_0\,.$$

The left-hand side reduces to $81 - (2)(1)(-1)(-5) = 71$, while the expression on the right is exactly the interpolation formula for $p(16)$. Therefore we have $p(16) = 71$, ten less than the intuitive guess of 81.

PROBLEMS

Faire parvenir les propositions de problèmes et les solutions à Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's (Terre-Neuve), Canada, A1C 5S7. Les propositions de problèmes doivent être accompagnées d'une solution ainsi que de références et d'autres indications qui pourraient être utiles à la rédaction. Si vous envoyez une proposition sans solution, vous devez justifier une solution probable en fournissant suffisamment d'information. Un numéro suivi d'une astérisque ($\star$) *indique que le problème a été proposé sans solution.*

Nous sollicitons en particulier des problèmes originaux. Cependant, d'autres problèmes intéressants pourraient être acceptables s'ils ne sont pas trop connus et si leur provenance est précisée. Normalement, si l'auteur d'un problème est connu, il faut demander sa permission avant de proposer un de ses problèmes.

Pour faciliter l'étude de vos propositions, veuillez taper ou écrire à la main (lisiblement) chaque problème sur une feuille distincte de format $8\frac{1}{2}$*"*$\times$**11***" ou* A4*, la signer et la faire parvenir au rédacteur en chef. Les propositions devront lui parvenir au plus tard le* **1er octobre 2002***. Vous pouvez aussi les faire parvenir par courriel à* **crux-editors@cms.math.ca**. (*Nous apprécierions de recevoir les problèmes et solutions envoyés par courriel au format* LaTeX). *Les fichiers graphiques doivent être de format « epic » ou « eps » (encapsulated postscript). Les solutions reçues après la date ci-dessus seront prises en compte s'il reste du temps avant la publication. Veuillez prendre note que nous n'acceptons pas les propositions par télécopieur.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

2713. *Proposé par Toshio Seimiya, Kawasaki, Japan.*

Soit O un point intérieur du triangle ABC et supposons que AO, BO et CO coupent BC, CA et AB en D, E et F, respectivement. Soit H le pied de la perpendiculaire à EF passant par D.

Montrer que les pieds des perpendiculaires à AF, FO, OE et EA passant par H sont sur un même cercle.

. . . .

Suppose that O is an interior point of $\triangle ABC$, and that AO, BO and CO meet BC, CA and AB at D, E and F, respectively. Let H be the foot of the perpendicular from D to EF.

Prove that the feet of the perpendiculars from H to AF, FO, OE and EA are concyclic.

2714. *Proposé par Toshio Seimiya, Kawasaki, Japan.*

Soient respectivement D et E des points sur les côtés AC et AB d'un triangle ABC, de sorte que $\angle DBC = 2\angle ABD$ et $\angle ECB = 2\angle ACE$. Si BD et CE se coupent en O et si $OD = OE$, que peut-on dire de ce triangle ?

. . . .

Suppose that D and E are points on sides AC and AB, respectively, of $\triangle ABC$, such that $\angle DBC = 2\angle ABD$ and $\angle ECB = 2\angle ACE$. Suppose that BD and CE meet at O, and that $OD = OE$. Characterize $\triangle ABC$.

2715. *Proposé par Toshio Seimiya, Kawasaki, Japan.*

Soit O le centre du cercle inscrit à un quadrilatère convexe $ABCD$. Si E et F sont respectivement les centres des cercles inscrits aux triangles ABC et ADC, montrer que A, O et le centre du cercle circonscrit au triangle AEF sont sur une droite.

. . . .

Suppose that the convex quadrilateral $ABCD$ has an incircle with centre O. Let E and F be the incentres of $\triangle ABC$ and $\triangle ADC$, respectively. Prove that A, O and the circumcentre of $\triangle AEF$ are collinear.

2716. *Proposé par Toshio Seimiya, Kawasaki, Japan.*

On suppose que

1. $ABCD$ est un quadrilatère convexe donné,
2. P est un point sur AD situé au-delà de A de telle sorte que $\angle APB = \angle BAC$,
3. Q est un point sur AD situé au-delà de D de telle sorte que $\angle DQC = \angle BDC$, et
4. $AP = DQ$.

Que peut-on dire du quadrilatère $ABCD$?

. . . .

Suppose that

1. convex quadrilateral $ABCD$ is given,
2. P is a point of AD produced beyond A such that $\angle APB = \angle BAC$,
3. Q is a point on AD produced beyond D such that $\angle DQC = \angle BDC$, and
4. $AP = DQ$.

Characterize quadrilateral $ABCD$.

2717. *Proposé par Mihály Bencze, Brasov, Romania.*
Pour tout triangle ABC, montrer que

$$8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \;\le\; \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right)\,.$$

....

For any triangle ABC, prove that

$$8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \;\le\; \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right)\,.$$

2718. *Proposé par Mihály Bencze, Brasov, Romania.*
Soit $A_k \in M_m(\mathbb{R})$ tels que $A_iA_j = O_m$, i, $j \in \{1, 2, \ldots, n\}$, avec $i < j$ et $x_k \in \mathbb{R}^*$, $(k = 1, 2, \ldots, n)$. Montrer que

$$\det\left(I_m + \sum_{k=1}^{n}\left(x_kA_k + x_k^2A_k^2\right)\right) \;\ge\; 0\,.$$

....

Let $A_k \in M_m(\mathbb{R})$ with $A_iA_j = O_m$, i, $j \in \{1, 2, \ldots, n\}$, with $i < j$ and $x_k \in \mathbb{R}^*$, $(k = 1, 2, \ldots, n)$. Prove that

$$\det\left(I_m + \sum_{k=1}^{n}\left(x_kA_k + x_k^2A_k^2\right)\right) \;\ge\; 0\,.$$

2719. *Proposé par Antal E. Fekete, Memorial University, St. John's, Newfoundland.*
Soit $k \in \mathbb{Z}$ et $n \in \mathbb{N}$. Montrer que, pour $n = 1$ et $n = 2$,

$$\sum_{j=0}^{\infty}\frac{(k+j)^n}{j!} \;=\; (-1)^n\sum_{j=0}^{\infty}\frac{(-k-2+j)^n}{j!}$$

et

$$\sum_{j=0}^{\infty}(-1)^j\frac{(k+j)^n}{j!} \;=\; (-1)^n\sum_{j=0}^{\infty}(-1)^j\frac{(-k-2+j)^n}{j!}$$

Ces égalités sont-elles vraies ou fausses pour d'autres valeurs entières positives de n ?

....

Let $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. Show that

$$\sum_{j=0}^{\infty}\frac{(k+j)^n}{j!} \;=\; (-1)^n\sum_{j=0}^{\infty}\frac{(-k-2+j)^n}{j!}$$

and

$$\sum_{j=0}^{\infty}(-1)^j\frac{(k+j)^n}{j!} = (-1)^n\sum_{j=0}^{\infty}(-1)^j\frac{(-k-2+j)^n}{j!}$$

for $n = 1$ and $n = 2$.

Are these equalities true or false for other positive integral values of n?

2720. *Proposé par Antal E. Fekete, Memorial University, St. John's, Newfoundland.*

Soit k un entier et n un entier non négatif.

(a) Montrer que $\sum\limits_{j=0}^{\infty}\frac{(k+j)^n}{j!}$ est un multiple entier de e et trouver la somme.

(b) Montrer que $\sum\limits_{j=0}^{\infty}(-1)^j\frac{(k+j)^n}{j!}$ est un multiple entier de $\frac{1}{e}$ et trouver la somme.

. . . .

Let k be an integer and n be a non-negative integer.

(a) Show that $\sum\limits_{j=0}^{\infty}\frac{(k+j)^n}{j!}$ is an integral multiple of e, and find the sum.

(b) Show that $\sum\limits_{j=0}^{\infty}(-1)^j\frac{(k+j)^n}{j!}$ is an integral multiple of $\frac{1}{e}$, and find the sum.

2721. *Proposé par Vedula N. Murty, Visakhapatnam, India.*

On considère l'équation cubique $x^3 - 19x + 30 = 0$. Il est facile de vérifier que les racines de cette équation sont -5, 2 et 3. Si l'on essaie de résoudre l'équation ci-dessus par la méthode trigonométrique, on trouve que les racines sont :

$$-2\rho^{\frac{1}{3}}\cos\left(\frac{\theta}{3}\right), \quad -2\rho^{\frac{1}{3}}\cos\left(\frac{2\pi+\theta}{3}\right), \quad \text{et} \quad -2\rho^{\frac{1}{3}}\cos\left(\frac{4\pi+\theta}{3}\right),$$

où $\rho^{\frac{1}{3}} = \sqrt{\frac{19}{3}}$ et $\cos\theta = 15\left(\frac{3}{19}\right)^{\frac{3}{2}}$.

Sans utiliser de calculatrice, montrer que

$$-2\rho^{\frac{1}{3}}\cos\left(\frac{\theta}{3}\right) = -5, \; -2\rho^{\frac{1}{3}}\cos\left(\frac{2\pi+\theta}{3}\right) = 2, \; \text{et} \; -2\rho^{\frac{1}{3}}\cos\left(\frac{4\pi+\theta}{3}\right) = 3\,.$$

Consider the cubic equation $x^3 - 19x + 30 = 0$. It is easily verified that the roots of this equation are -5, 2 and 3. If one tries to solve the above equation using trigonometry, the roots come out as

$$-2\rho^{\frac{1}{3}}\cos\left(\frac{\theta}{3}\right), \quad -2\rho^{\frac{1}{3}}\cos\left(\frac{2\pi+\theta}{3}\right), \quad \text{and} \quad -2\rho^{\frac{1}{3}}\cos\left(\frac{4\pi+\theta}{3}\right),$$

where $\rho^{\frac{1}{3}} = \sqrt{\frac{19}{3}}$ and $\cos\theta = 15\left(\frac{3}{19}\right)^{\frac{3}{2}}$.

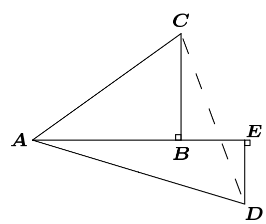
Show, without the use of a calculator, that

$$-2\rho^{\frac{1}{3}}\cos\left(\frac{\theta}{3}\right) = -5,\ -2\rho^{\frac{1}{3}}\cos\left(\frac{2\pi+\theta}{3}\right) = 2,\ \text{and}\ -2\rho^{\frac{1}{3}}\cos\left(\frac{4\pi+\theta}{3}\right) = 3\,.$$

2722. *Proposé par Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

On considère deux triangles pythagoriciens comme indiqué dans la figure. Les longueurs de AC, CB, AD et DE sont des nombres impairs et les longueurs des côtés superposés, AE et AB, sont des nombres pairs.

Existe-t-il une configuration semblable telle que la longueur de CD soit un entier ?



. . . .

Consider two Pythagorean triangles as indicated in the figure. The lengths of AC, CB, AD and DE are odd integers and the lengths of the overlapping sides, AE and AB, are even integers.

Does there exist such a configuration of Pythagorean triangles such that the length of CD is an integer?

2723. *Proposé par Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Soit n_1, n_2, $\ldots$, n_k $(1 \leq k \leq N)$ des entiers non négatifs tels que $n_1 + n_2 + \cdots + n_k = N$. Trouver la valeur minimale de la somme $\sum\limits_{j=1}^{k} \binom{n_j}{m}$ lorsque (a) $m = 2$; (b)* $m \geq 3$.

. . . .

For $1 \leq k \leq N$, let n_1, n_2, $\ldots$, n_k be non-negative integers such that $n_1 + n_2 + \cdots + n_k = N$. Determine the minimum value of the sum $\sum\limits_{j=1}^{k} \binom{n_j}{m}$ when (a) $m = 2$; (b)* $m \geq 3$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2605. [2001 : 49] *Proposed by K.R.S. Sastry, Bangalore, India.*

In triangle ABC, with median AD and internal angle bisector BE, we are given $AB = 7$, $BC = 18$ and $EA = ED$. Find AC.

I. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let M, N be feet of the perpendiculars from A, E to BC, respectively. Let J be on BN such that $BJ = BA = 7$. Then $\triangle ABE \cong \triangle JBE$ by SAS. Therefore, $EJ = EA = ED$. Furthermore, $JN = ND = 1$ and $NC = 10$. Also $AE/EC = AB/BC = 7/18$. Hence

$$MC \;=\; NC \times \frac{AC}{EC} \;=\; 10 \times \frac{25}{18} \;=\; \frac{125}{9}\,.$$

Thus

$$BM \;=\; 18 - \frac{125}{9} = \frac{37}{9}.$$

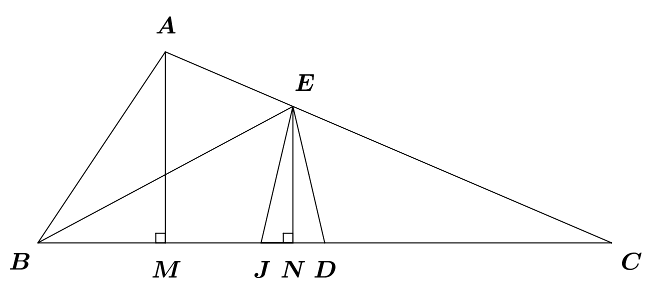
By the Theorem of Pythagoras:

$$AC^2 - AB^2 \;=\; (MC^2 + AM^2) - (MB^2 + AM^2) \;=\; MC^2 - MB^2\,.$$

Therefore,

$$\begin{aligned} AC^2 \;&=\; AB^2 + \left(\frac{125}{9}\right)^2 - \left(\frac{37}{9}\right)^2 \\ &=\; 49 + \frac{(125+37)(125-37)}{81} = 49 + \frac{162 \cdot 88}{81} \;=\; 225\,, \end{aligned}$$

from which we conclude that $AC = 15$.



II. *Solution by Henry Liu, student, University of Memphis, TN, USA.*

Let J lie on side BC with $BJ = 7$. Then (as in solution I above) $\triangle ABE \cong \triangle JBE$. Thus, $\angle BAE = \angle BJE$. Also, $ED = EA = EJ$, which implies that $\triangle JDE$ is isosceles, whence $\angle CDE = \angle BJE = \angle BAE$. Therefore, $\triangle CDE$ and $\triangle CAB$ are similar. Set $y = EA = ED$. Then

$$\begin{array}{rrcll} & \dfrac{9}{y} & = & \dfrac{AC}{7} & \Longrightarrow \quad AC \cdot y \;=\; 63 \\ \text{and} & \dfrac{AC-y}{9} & = & \dfrac{18}{AC} & \Longrightarrow \quad AC^2 - AC\cdot y \;=\; 162 \\ & & & & \Longrightarrow \quad AC^2 \;=\; 162+63 \;=\; 225 \\ & & & & \Longrightarrow \quad AC \;=\; 15\,. \end{array}$$

III. *Solution by Paul Jeffreys, student, Berkhamsted Collegiate School, UK.*

Since BE bisects $\angle ABC$, we have $7/AE = 18/EC$. Thus

$$\frac{AE}{EC} = \frac{7}{18} \qquad \text{and} \qquad EC = \frac{18}{25}AC. \tag{1}$$

Let $\alpha = \angle BAC$ and $\beta = \angle ABC$. By the Law of Sines on $\triangle ABE$, we have

$$\frac{AE}{\sin(\beta/2)} = \frac{BE}{\sin\alpha} \qquad \Longleftrightarrow \qquad AE\sin\alpha = BE\sin(\beta/2).$$

Applying the Law of Sines to $\triangle BDE$ yields:

$$\frac{DE}{\sin(\beta/2)} = \frac{BE}{\sin(\angle BDE)} \quad \Longleftrightarrow \quad DE\sin(\angle BDE) = BE\sin(\beta/2)\,.$$

Since $AE = ED$ we have $\sin(\angle BDE) = \sin\alpha$. If $\angle BDE = \alpha$, then $\triangle BAE$ is similar to $\triangle BDE$. Since $BD = 9 \neq 7 = BA$, we get a contradiction, whence we have $\angle BDE = 180^\circ - \alpha$. Thus $BDEA$ is a cyclic quadrilateral. Since $\angle BDE = 180^\circ - \alpha$, we also have $\angle CDE = \alpha$, which implies that $\triangle CDE$ is similar to $\triangle CAB$. Therefore, $9/AC = EC/18$. From (1), we have

$$\frac{9}{AC} = \frac{18}{25}\cdot\frac{AC}{18}\,, \quad \text{so that} \quad 9\cdot 25 = AC^2\,.$$

Thus, $AC = 15$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO TEAM 2001; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SCOTT BROWN, Auburn University, Montgomery, AL; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES DIMINNIE and KARL HAVLAK, Angelo State University, San Angelo, TX; C. FESTRAETS-HAMOIR, Brussels, Belgium; VINAYAK GANESHAN, student, University of Waterloo, Waterloo, Ontario; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; DAVID LOEFFLER, *student, Cotham School, Bristol, UK; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; REVAÌ MATH CLUB, Győr, Hungary; JOEL SCHLOSBERG, student, New York University, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany (2 solutions); ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; KENNETH M. WILKE, Topeka, KS, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.*

Most solvers used a variation of solution III above. Kandall observes "It is easy to show that if $a = 2c$, then $EA = ED$ regardless of the length of AC", where a and c, of course, are the lengths of the sides opposite A and C, respectively.

Kandall, Konečný, and Seiffert all consider the more general problem with $BC = a$, $AB = c$ and $DE = EA$, and conclude that $AC = \sqrt{\frac{1}{2}a(a+c)}$.

2607. [2001 : 49] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

(a) Suppose that $q > p$ are odd primes such that $q = pn+1$, where n is an integer greater than 1. Let z be a complex number such that $z^q = 1$.

Prove that $\dfrac{z^p-1}{z^p+1} = \displaystyle\sum_{j=1}^{q-1}(-1)^{\left\lfloor\frac{j-1}{p}\right\rfloor} z^j$.

(b) Suppose that $q > 3$ is an odd prime such that $q = 3n+2$, where n is an integer greater than 1. Let z be a complex number such that $z^q = 1$.

Prove that $\dfrac{z^3-1}{z^3+1} = \displaystyle\sum_{j=1}^{q-1}(-1)^{\left\lfloor\frac{j-3}{3}\right\rfloor} z^j$.

Solution by Michel Bataille, Rouen, France.

(a) Observing that n is necessarily even, we may write

$$\begin{aligned}\sum_{j=1}^{q-1}(-1)^{\left\lfloor\frac{j-1}{p}\right\rfloor} z^j &= \sum_{j=1}^{pn}(-1)^{\left\lfloor\frac{j-1}{p}\right\rfloor} z^j \\ &= \sum_{j=1}^{p} z^j - \sum_{j=1}^{p} z^{j+p} + \sum_{j=1}^{p} z^{j+2p} - \cdots - \sum_{j=1}^{p} z^{j+(n-1)p}.\end{aligned}$$

If $z = 1$, this sum is $p - p + p - p + \cdots + p - p = 0$. Note that $\dfrac{z^p-1}{z^p+1} = 0$ as well.

If $z \neq 1$, the sum is

$$\begin{aligned}& z\frac{z^p-1}{z-1} - z^{p+1}\frac{z^p-1}{z-1} + \cdots - z^{(n-1)p+1}\frac{z^p-1}{z-1} \\ &\quad = \frac{z^p-1}{z-1}(z)\left(1 - z^p + z^{2p} - \cdots - z^{(n-1)p}\right) \\ &\quad = \frac{z^p-1}{z-1}(z)\frac{(-z^p)^n-1}{(-z)^p-1} = \frac{z^p-1}{z^p+1}\,\frac{(-z)(z^{q-1}-1)}{z-1} = \frac{z^p-1}{z^p+1}\end{aligned}$$

(the latter, using $z^{q-1} = \frac{1}{z}$). Thus, we have (a). Note that it is not necessary to suppose that p and q are primes — it is sufficient to require that p and q are odd.

(b) This time, n is necessarily odd. We have

$$\begin{aligned} S &= \sum_{j=1}^{q-1}(-1)^{\lfloor\frac{j-3}{3}\rfloor} z^j \;=\; \sum_{j=1}^{3n+1}(-1)^{\lfloor\frac{j-3}{3}\rfloor} z^j \\ &= -z - z^2 + \left(z^3+z^4+z^5\right) - \left(z^6+z^7+z^8\right) + \cdots \\ &\qquad - \left(z^{3(n-1)} + z^{3(n-1)+1} + z^{3(n-1)+2}\right) + z^{3n} + z^{3n+1} \\ &= -z - z^2 + z^{3n} + z^{3n+1} \\ &\qquad + z^3\left(1+z+z^2\right)\left(1 - z^3 + z^6 - \cdots - z^{3(n-2)}\right). \end{aligned}$$

If $z = 1$, we have $S = 0 = \dfrac{z^3-1}{z^3+1}$.

If $z \neq 1$, we have

$$S = -z - z^2 + z^{q-2} + z^{q-1} + \frac{z^3-1}{z-1}\, z^3\, \frac{1-(-z^3)^{n-1}}{1+z^3}\,.$$

Using $z^{q-2} = \frac{1}{z^2}$, $z^{q-1} = \frac{1}{z}$ and $z^{3n-3} = \frac{1}{z^5}$, we obtain

$$S = \frac{z^3-1}{z^3+1}\left(\frac{-(z+1)(z^3+1)}{z^2} + \frac{z^3}{z-1}\left(1 - \frac{1}{z^5}\right)\right) = \frac{z^3-1}{z^3+1}\,.$$

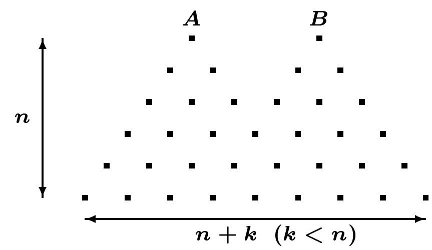
This completes the proof of (b). Similarly to (a), it is sufficent to require that q is odd.

Also solved by AUSTRIAN IMO–TEAM 2001; VINAYAK GANESHAN, student, University of Waterloo, Walerloo, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Cotham School, Bristol, UK; JOEL SCHLOSBERG, student, New York University, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

In fact, only Janous and Seiffert considered the case: $z = 1$. The other solvers ignored it. The editor was generous in not classing their solutions as incomplete.

2612★. [2001 : 49] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Two "Galton"–figures are given as follows:



(There are n levels in total; there are k levels such that there is no "intersection" between the levels emanating from A and B.)

Let two balls start at the same time from A and B. Each ball moves either $\swarrow$ or $\searrow$ with probability $\frac{1}{2}$.

Determine the probability $P(n,k)$ $(1 \leq k < n)$ such that the two balls reach the bottom level without colliding.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Denote by $c(n,k)$, the number of pairs of colliding paths. Then, $c(n,k) = 0$ if $k \geq n$, $c(n, n-1) = 1$, and $c(n,0) = 2^{n-1} \cdot 2^{n-1} = 2^{2n-2}$. Also, there are $c(n-1,k)$ pairs of colliding paths starting with $(\swarrow, \swarrow)$, $c(n-1,k+1)$ with $(\swarrow, \searrow)$, $c(n-1,k-1)$ with $(\searrow, \swarrow)$, and $c(n-1,k)$ with $(\searrow, \searrow)$. Therefore,

$$c(n,k) \;=\; c(n-1,k-1) + 2c(n-1,k) + c(n-1,k+1)\,.$$

We claim that $c(n,k) \;=\; \binom{2n-2}{n-k-1} + 2\sum\limits_{i=1}^{n-k-2}\binom{2n-2}{i}$.

Indeed, using the identity $\binom{m}{j} \;=\; \binom{m-2}{j-2} + 2\binom{m-2}{j-1} + \binom{m-2}{j}$, we see easily that it satisfies the recursion. Moreover, since

$$\binom{2n-2}{n-1} + 2\sum_{i=0}^{n-2}\binom{2n-2}{i} \;=\; \sum_{i=0}^{2n}\binom{2n-2}{i} \;=\; 2^{2n-2}\,,$$

we see that all three boundary conditions are satisfied as well. Hence,

$$P(n,k) \;=\; 1 - \frac{c(n,k)}{2^{2n-2}} \;=\; \frac{2^{2n-2} - c(n,k)}{2^{2n-2}} \;=\; \frac{1}{2^{2n-2}}\sum_{i=n-k}^{n+k-1}\binom{2n-2}{i}\,.$$

Also solve by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; KEITH EKBLAW, Walla Walla, WA, USA; RICHARD I. HESS, Rancho Palos

Verdes, CA, USA; ERIC POSTPISCHIL, Nashua, NH, USA; and JOEL SCHLOSBERG, student, New York University, NY, USA. There was one incomplete solution.

The submitted solutions varied in length from the above to one of ten pages. Postpischil noted that, in the CRC Concise Encyclopedia of Mathematics*, CRC Press, Boca Raton, FL, USA* (1999)*, p. 138, Eric Weisstein shows how to write the sum in terms of the Beta Function* (*also known as the Eulerian Integral of the Second Kind*) *and the incomplete Beta Function.*

2613. [2001 : 136] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.* Dedicated to Toshio Seimiya, and based on his problem 2515.

In $\triangle ABC$, the three cevians AD, BE and CF through a non-exterior point P are such that $AF + BD + CE = s$ (the semi-perimeter). Characterize $\triangle ABC$ for each of the cases when P is (i) the orthocentre, and (ii) the Lemoine point.

[Ed. The Lemoine point is also known as the symmedian point. See, for example, James R. Smart, *Modern Geometries*, 4^{th} Edition, 1994, Brooks/Cole, California, USA. p. 161.]

Solution by Theoklitos Paragiou, Limassol, Cyprus, Greece.

In both cases $\triangle ABC$ is isosceles.

(i) When P is the orthocentre, AD, CF, and BE are the altitudes. We have therefore, $BD = c\cos B = \dfrac{c^2+a^2-b^2}{2a}$, and likewise, $CD = \dfrac{a^2+b^2-c^2}{2b}$, and $AF = \dfrac{b^2+c^2-a^2}{2c}$. Thus, the following statements are equivalent to the given condition:

$$AF + BD + CE \;=\; s\,,$$

$$\frac{b^2+c^2-a^2}{2c}+\frac{c^2+a^2-b^2}{2a}+\frac{a^2+b^2-c^2}{2b} \;=\; \frac{a+b+c}{2}\,,$$

$$ab(b^2-a^2)+bc(c^2-b^2)+ac(a^2-c^2) \;=\; 0\,,$$

$$(a-b)(b-c)(c-a)(a+b+c) \;=\; 0\,.$$

The last equation holds if and only if the given triangle is isosceles.

(ii) Standard references tell us that when P is the Lemoine point,

$$AF \;=\; \frac{cb^2}{a^2+b^2}\,,\quad BD \;=\; \frac{ac^2}{b^2+c^2}\,,\quad \text{and}\quad CE \;=\; \frac{ba^2}{c^2+a^2}\,.$$

Thus, the following statements are equivalent to the given condition:

$$AF + BD + CE \;=\; s\,,$$

$$\frac{cb^2}{a^2+b^2}+\frac{ac^2}{b^2+c^2}+\frac{ba^2}{c^2+a^2} \;=\; \frac{a+b+c}{2}\,,$$

$$\frac{(a-b)(b-c)(c-a)[a^3(b+c)+2a^2bc+a(b^3+2b^2c+2bc^2+c^3)+bc(b^2+c^2)]}{2(a^2+b^2)(b^2+c^2)(c^2+a^2)} = 0\,.$$

Once again, the last equation holds if and only if the given triangle is isosceles.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JOEL SCHLOSBERG, student, New York University, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and by the proposer.

Most solvers used essentially the same argument as in the featured solution, sometimes with more detail, sometimes with less. Together with problem 2515 [2000 : 114; 2001 : 144], *we now have established that*

$$AF + BD + CE \;=\; s \quad \textit{if and only if} \quad \triangle ABC \textit{ is isosceles}$$

when P is the incentre, the orthocentre, and the Lemoine point. David Loeffler continues the theme with the comment:

In fact, huge numbers of triangle centres may be dealt with in the same way, if you have a computer algebra system or incredible patience! The Mittenpunkt — Kimberling's X(9) [Clark Kimberling, Encyclopedia of Triangle Centers, `http://cedar.evansville.edu/~ck6/tcenters/`; or *Math. Mag.* **67**:3 (June 1994), 163–187] — *satisfies the condition only for isosceles triangles, although this is slightly more tricky to prove than the above. The same is true of the Spieker center X(10), all of the power points, and various others such as X(37), X(38), X(39), X(42), and X(43).*

Klamkin wondered whether the same is true of the circumcentre. However, before we get carried away with wild conjectures note that $AF + BD + CE = s$ for **all** *triangles when P is either the centroid or the Gergonne point.*

2614. [2001 : 136] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.* Dedicated to Toshio Seimiya, and suggested by his problem 2514.

In $\triangle ABC$, the two cevians through a non-exterior point P meet AC and AB at D and E, respectively. Suppose that $AE = BD$ and $AD = CE$. Characterize $\triangle ABC$ for the cases when P is (i) the orthocentre, (ii) the centroid, and (iii) the Lemoine point.

Solution by Henry Liu, student, University of Memphis, Memphis, TN, USA.

(i) When P is the orthocentre, BD and CE are the altitudes. By SAS and the given conditions, $\triangle ADB \cong \triangle CEA$, so that $AB = AC$ and consequently, $\angle ABC = \angle ACB$. Further, $\triangle BEC \cong \triangle CDB$ since they are similar right triangles that share their hypotenuse BC. Thus, $BD = CE = AD$. Therefore. $\triangle ABC$ is isosceles with $A = 45^\circ$ and $B = C = 67.5^\circ$.

(ii) When P is the centroid, E and D are mid-points of their respective sides. By SSS, $\triangle BEC \cong \triangle BDC$. Since these triangles have BC in common while the vertices D and E lie on the same side of the line BC, the two triangles must coincide (with E and D the same vertex). Since AB

contains E and AC contains D, the two sides of $\triangle ABC$ coincide, and the triangle is therefore degenerate (with $\angle A = 0^\circ$ and the vertices B and C coincident).

(iii) As in problem 2613, when P is the Lemoine point,

$$AE \;=\; \frac{cb^2}{a^2+b^2} \quad \text{and} \quad EB \;=\; \frac{ca^2}{a^2+b^2}\,.$$

By Stewart's Theorem we have $c(CE^2 + AE \cdot EB) = a^2 AE + b^2 EB$, so that $c\left(CE^2 + \left(\frac{abc}{a^2+b^2}\right)^2\right) = \frac{2a^2b^2c}{a^2+b^2}$ and therefore,

$$CE^2 \;=\; a^2b^2\left(\frac{2a^2+2b^2-c^2}{(a^2+b^2)^2}\right)\,.$$

Similarly, we have

$$AD \;=\; \frac{bc^2}{a^2+c^2} \quad \text{and} \quad BD^2 = a^2c^2\left(\frac{2a^2+2c^2-b^2}{(a^2+c^2)^2}\right)\,.$$

The given conditions imply $\left(\frac{AE}{AD}\right)^2 \;=\; \left(\frac{BD}{CE}\right)^2$, so that

$$\left(\frac{cb^2(a^2+c^2)}{bc^2(a^2+b^2)}\right)^2 \;=\; \frac{a^2c^2}{a^2b^2}\cdot\frac{(a^2+b^2)^2}{(a^2+c^2)^2}\cdot\frac{2a^2+2c^2-b^2}{2a^2+2b^2-c^2}\,. \tag{1}$$

Set $x = a^2$, $y = b^2$, $z = c^2$, and (1) reduces to

$$y^2(2x+2y-z)(x+z)^4 - z^2(2x+2z-y)(x+y)^4 \;=\; 0\,.$$

Since $y = z$ satisfies the equation, we can factor out $(y-z)$ to get

$$(y-z)(y^4z^2 - y^3z^3 + y^2z^4 + P(x,y,z)) \;=\; 0\,,$$

where $P(x,y,z)$ is a polynomial in x, y, and z having all its terms positive. Note that the first three terms in the factor on the right satisfy

$$y^4z^2 - y^3z^3 + y^2z^4 \;=\; y^2z^2((y-z)^2 + yz) > 0\,.$$

Since $y - z = (b-c)(b+c)$, we see that (1) is equivalent to

$$(b-c)Q(a,b,c) \;=\; 0\,,$$

where Q is a polynomial that is positive for all positive values of a, b, and c. We conclude that $b = c$, so that $\triangle ABC$ is isosceles.

It remains to determine the shape of $\triangle ABC$. [Here the editor has replaced Liu's argument with a shorter calculation.]

Instead of (1), use $\left(\frac{AE}{CE}\right)^2 \;=\; \left(\frac{BD}{AD}\right)^2$, and set $b = c$ to get

$$b^4 \;=\; a^2(2a^2+b^2)\,.$$

Since a, $b > 0$, this equation implies that $b = \sqrt{2}a$. Therefore, $\triangle ABC$ must satisfy $b = c = \sqrt{2}a$ with $\angle A = \cos^{-1}\left(\frac{3}{4}\right)$. One easily checks that these conditions are also sufficient.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JOEL SCHLOSBERG, student, New York University, NY, USA ((i) *and* (ii) *only*)*; PETER Y. WOO, Biola University, La Mirada, CA, USA* ((i) *and* (ii) *only*)*; and the proposer.*

2615. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Suppose that x_1, x_2, ..., x_n, are non-negative numbers such that

$$\sum x_1^2 + \sum (x_1 x_2)^2 = \frac{n(n+1)}{2},$$

where the sums here and subsequently are symmetric over the subscripts 1, 2, ..., n.

(a) Determine the maximum of $\sum x_1$.

(b)★ Prove or disprove that the minimum of $\sum x_1$ is $\sqrt{\dfrac{n(n+1)}{2}}$.

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

(a) The given equation,

$$\sum_{k=1}^{n} x_k^2 + \sum_{1 \le j < k \le n} (x_j x_k)^2 = \frac{n(n+1)}{2}, \tag{1}$$

is satisfied when $x_1 = x_2 = \ldots = x_n = 1$, so that the maximum of $\sum x_1$ is greater than or equal to n. Using the trivial inequality $2t - t^2 \le 1$, $t \in R$, we obtain

$$\begin{aligned}
\left(\sum x_1\right)^2 &= \left(\sum_{k=1}^{n} x_k\right)^2 = \sum_{k=1}^{n} x_k^2 + 2\sum_{1 \le j < k \le n} x_j x_k \\
&= \sum_{k=1}^{n} x_k^2 + 2\sum_{1 \le j < k \le n} x_j x_k + \sum_{1 \le j\ <\ k \le n} (x_j x_k)^2 - (x_j x_k)^2 \\
&= \frac{n(n+1)}{2} + \sum_{1 \le j < k \le n} 2x_j x_k - (x_j x_k)^2 \\
&\le \frac{n(n+1)}{2} + \sum_{1 \le j < k \le n} 1 = n^2,
\end{aligned}$$

or $\sum x_1 \le n$. Hence, the maximum of $\sum x_1$ is n.

(b) Clearly, if $n = 1$, then the minimum of $\sum x_1$ is 1.

Let $n = 2$. From the condition $x_1^2 + x_2^2 + (x_1x_2)^2 = 3$, we have $x_1x_2 \leq \sqrt{3} < 2$, so that $(x_1 + x_2)^2 = 3 + x_1x_2(2 - x_1x_2) \geq 3$, or $x_1 + x_2 \geq \sqrt{3}$. Since an equality is attained when $x_1 = \sqrt{3}$ and $x_2 = 0$, it follows that the minimum of $\sum x_1$ is $\sqrt{3}$.

Let $n = 3$. First, suppose that $\max\{x_1x_2, x_1x_3, x_2x_3\} > 2$. If $x_1x_2 > 2$, then $x_1 + x_2 + x_3 \geq x_1 + x_2 \geq 2\sqrt{x_1x_2} > 2\sqrt{2} > \sqrt{6}$. Otherwise, we have $(x_1+x_2+x_3)^2 = 6+x_1x_2(2-x_1x_2)+x_1x_3(2-x_1x_3)+x_2x_3(2-x_2x_3) \geq 6$, or $x_1 + x_2 + x_3 \geq \sqrt{6}$. Since an equality is attained when $x_1 = \sqrt{6}$ and $x_2 = x_3 = 0$, we conclude that the minimum of $\sum x_1$ is $\sqrt{6}$.

We choose to disprove for $n \geq 4$. The equation (1) is satisfied when $x_1 = x_2 = \sqrt{\frac{1}{2}\left(\sqrt{2n(n+1)+4} - 2\right)}$ and $x_3 = x_4 = \cdots = x_n = 0$. However, $\sum x_1 = \sqrt{2\left(\sqrt{2n(n+1)+4} - 2\right)} < \sqrt{\frac{1}{2}n(n+1)}$, because the last inequality easily reduces to $n(n+1) > 16$, which is true for $n \geq 4$.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA. Part (a) *only was solved by the proposer. One solver sent an incomplete solution. Another solver misinterpreted the condition and solved a different problem.*

2617. [2001 : 137] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

A problem in one book was to prove that each edge of an isosceles tetrahedron is equally inclined to its opposite edge. A problem in another book was to prove that the three angles formed by the opposite edges of a tetrahedron cannot be equal unless they are at right angles.

1. Show that only the second result is valid.
2. Show that a tetrahedron which is both isosceles and orthocentric must be regular.

Solution by Joel Schlosberg, student, New York University, NY, USA.

If the three edges that come out of a single vertex of the tetrahedron are labelled as the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, then the six sides of the tetrahedron are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{c}-\mathbf{b}$, $\mathbf{b}-\mathbf{a}$, $\mathbf{a}-\mathbf{c}$.

If X, Y and Z are the three angles formed by the opposite edges, then

$$\cos X = \pm\frac{\mathbf{a}\cdot(\mathbf{c}-\mathbf{b})}{|\mathbf{a}||\mathbf{c}-\mathbf{b}|} = \pm\frac{\mathbf{a}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{c}-\mathbf{b}|},$$

$$\cos Y = \pm\frac{\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}||\mathbf{a}-\mathbf{c}|}, \qquad \cos Z = \pm\frac{\mathbf{b}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}}{|\mathbf{c}||\mathbf{b}-\mathbf{a}|}.$$

(The "angle between the opposite segments" is a slightly ambiguous phrase which could refer to either of the two supplementary angles. However, this does not affect the problem, since the [absolute values of the] cosines of two supplementary angles are the same.)

1. A counterexample to the first result is the tetrahedron with vertices $(0,0,0)$, $(0,2u,0)$, (u,u,v), $(-u,u,v)$, where $2u^2 \neq v^2$, so that $\mathrm{a} = (0,2u,0)$, $\mathrm{b} = (u,u,v)$ and $\mathrm{c} = (-u,u,v)$. A simple calculation shows that

$$\begin{aligned} |\mathrm{c}-\mathrm{b}| &= |(-2u,0,0)| = |(0,2u,0)| = |\mathrm{a}|\,, \\ |\mathrm{a}-\mathrm{c}| &= |(u,u,-v)| = |(u,u,v)| = |\mathrm{b}|\,, \\ |\mathrm{b}-\mathrm{a}| &= |(u,-u,v)| = |(-u,u,v)| = |\mathrm{c}|\,, \end{aligned}$$

so that the tetrahedron is isosceles. However,

$$\cos X = \pm\frac{\mathrm{a}\cdot\mathrm{c}-\mathrm{a}\cdot\mathrm{b}}{|\mathrm{a}||\mathrm{c}-\mathrm{b}|} = \pm\frac{2u^2-2u^2}{|\mathrm{a}|^2} = 0\,,$$

but

$$\cos Y = \pm\frac{\mathrm{a}\cdot\mathrm{b}-\mathrm{b}\cdot\mathrm{c}}{|\mathrm{b}||\mathrm{a}-\mathrm{c}|} = \pm\frac{2u^2-v^2}{|\mathrm{b}|^2} \neq 0\,.$$

Therefore $\cos X \neq \pm\cos Y$, and these angles are not equal.

To prove the second result, suppose that the three angles formed by opposite sides are equal. Without loss of generality, assume that $\mathrm{a}\cdot\mathrm{b} \leq \mathrm{a}\cdot\mathrm{c} \leq \mathrm{b}\cdot\mathrm{c}$. Then

$$|\cos X| = |\cos Y| = |\cos Z|\,,$$

and thus,

$$\frac{\mathrm{a}\cdot\mathrm{c}-\mathrm{a}\cdot\mathrm{b}}{|\mathrm{a}||\mathrm{c}-\mathrm{b}|} = \frac{\mathrm{b}\cdot\mathrm{c}-\mathrm{a}\cdot\mathrm{b}}{|\mathrm{b}||\mathrm{a}-\mathrm{c}|} = \frac{\mathrm{b}\cdot\mathrm{c}-\mathrm{a}\cdot\mathrm{c}}{|\mathrm{c}||\mathrm{b}-\mathrm{a}|}\,.$$

However, any equation of the form $D(x-y) = E(z-y) = F(z-x)$ has the unique solution $x = y = z$. (The equivalent homogeneous linear system $Dx + (E-D)y - Ez = 0$, $Fx - Ey + (E-F)z = 0$, has the solution set $x = y = z$; these are the only solutions, since any solution to a given n–equation, $(n+1)$–variable system of homogeneous linear equations is a scalar multiple of any other solution)

Therefore $\mathrm{a}\cdot\mathrm{b} = \mathrm{a}\cdot\mathrm{c} = \mathrm{b}\cdot\mathrm{c}$, so that $\mathrm{a}\cdot(\mathrm{c}-\mathrm{b}) = 0$, $\mathrm{b}\cdot(\mathrm{a}-\mathrm{c}) = 0$ and $\mathrm{c}\cdot(\mathrm{b}-\mathrm{a}) = 0$. Hence $\mathrm{a} \perp (\mathrm{c}-\mathrm{b})$, $\mathrm{b} \perp (\mathrm{a}-\mathrm{c})$ and $\mathrm{c} \perp (\mathrm{b}-\mathrm{a})$, implying that the opposite angles are right angles.

2. If a tetrahedron with sides a, b, c, $\mathrm{c}-\mathrm{b}$, $\mathrm{b}-\mathrm{a}$, $\mathrm{a}-\mathrm{c}$ is isosceles, then $|\mathrm{a}| = |\mathrm{c}-\mathrm{b}|$, $|\mathrm{b}| = |\mathrm{a}-\mathrm{c}|$ and $|\mathrm{c}| = |\mathrm{b}-\mathrm{a}|$. Therefore,

$$|\mathrm{a}|^2 = |\mathrm{c}-\mathrm{b}|^2 \quad \text{or} \quad \mathrm{a}\cdot\mathrm{a} = (\mathrm{c}-\mathrm{b})\cdot(\mathrm{c}-\mathrm{b}) = \mathrm{c}\cdot\mathrm{c}+\mathrm{b}\cdot\mathrm{b}-2\mathrm{b}\cdot\mathrm{c}\,,$$

and thus,

$$-\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{c} \;=\; 2\mathbf{b}\cdot\mathbf{c}\,.$$

Similarly

$$\mathbf{a}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{c} \;=\; 2\mathbf{a}\cdot\mathbf{c} \qquad\text{and}\qquad \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-\mathbf{c}\cdot\mathbf{c} = 2\mathbf{a}\cdot\mathbf{b}\,.$$

If the tetrahedron is orthocentric, then $\mathbf{a}\cdot(\mathbf{c}-\mathbf{b}) = \mathbf{b}\cdot(\mathbf{a}-\mathbf{c}) = \mathbf{c}\cdot(\mathbf{b}-\mathbf{a}) = 0$, and further, we have $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c}$. Combining this with the previous equations gives us

$$-\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{c} \;=\; \mathbf{a}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{c} \;=\; \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-\mathbf{c}\cdot\mathbf{c}\,.$$

Combining these equations pairwise allows us to deduce that $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{c}$, and hence that $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$. Combining this with the equations $|\mathbf{a}| = |\mathbf{c}-\mathbf{b}|$, $|\mathbf{b}| = |\mathbf{a}-\mathbf{c}|$ and $|\mathbf{c}| = |\mathbf{b}-\mathbf{a}|$ shows us that all six sides are equal in length, so that the tetrahedron is regular.

Also solved by JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2618 *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Determine a geometric problem whose solution is given by the positive solution of the equation

$$3x^2\left(\frac{1}{\sqrt{4R^2+x^2-a^2}}+\frac{1}{\sqrt{4R^2+x^2-b^2}}+\frac{1}{\sqrt{4R^2+x^2-c^2}}\right)$$
$$= \left(\sqrt{4R^2+x^2-a^2}+\sqrt{4R^2+x^2-b^2}+\sqrt{4R^2+x^2-c^2}+a+b+c\right),$$

where a, b, c and R are the sides and circumradius of a given triangle ABC.

Solution by the proposer.

We show that x is the altitude to the face ABC of an orthocentric tetrahedron of maximum isoperimetric quotient, $\frac{V}{E^3}$, where V and E are the volume and total edge length of the tetrahedron, respectively.

If $PABC$ is an orthocentric tetrahedron, then P must lie on a line through H, the orthocentre of ABC, and perpendicular to the plane of ABC. Then

$$PA \;=\; \sqrt{4R^2\cos^2 A+x^2} \;=\; \sqrt{4R^2+x^2-a^2}\,, \quad \text{etc.},$$

and $3V = x[ABC]$. The given equation corresponds to $\frac{d}{dx}\left(\frac{V}{E^3}\right) = 0$. That the maximum is unique follows by dividing both sides of the given equation by x, and then noting that the left hand side is an increasing function of x, whereas the right hand side is a decreasing one.

No other solutions were received.

2620. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta, dedicated to Murray S. Klamkin, on his 80th birthday.*

Three cards are handed to you. On each card are three non-negative real numbers, written one below the other, so that the sum of the numbers on each card is 1. You are permitted to put the three cards in any order you like, then write down the first number from the first card, the second number from the second card, and the third number from the third card. You add these three numbers together.

Prove that you can always arrange the three cards so that your sum lies in the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$. (Corrected)

Solution by Michel Bataille, Rouen, France.

Denote by (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) the triples of numbers respectively written on each of the cards. By hypothesis, these numbers are non-negative and

$$a_1 + a_2 + a_3 \;=\; b_1 + b_2 + b_3 \;=\; c_1 + c_2 + c_3 \;=\; 1\,.$$

Following the process described in the statement of the problem, we can form six sums, namely

$$\begin{aligned} s_1 &= a_1 + b_2 + c_3\,, & s_2 &= a_1 + c_2 + b_3\,, \\ s_3 &= b_1 + a_2 + c_3\,, & s_4 &= b_1 + c_2 + a_3\,, \\ s_5 &= c_1 + a_2 + b_3\,, & s_6 &= c_1 + b_2 + a_3\,. \end{aligned}$$

For the purpose of contradiction we will suppose that $s_i \notin \left[\frac{1}{2}, \frac{3}{2}\right]$ for $i = 1$, 2, $\ldots$, 6. Since $s_2 + s_3 + s_6 = 3$, one of the numbers s_2, s_3, s_6 must be at least 1, and one must be at most 1. Say, $s_2 \le 1$ and $s_6 \ge 1$. By supposition, we even have $s_2 < \frac{1}{2}$ and $s_6 > \frac{3}{2}$. The latter implies that we cannot have $s_3 > \frac{3}{2}$ (otherwise $s_3 + s_6 > 3$); hence $s_3 < \frac{1}{2}$. Now, $s_2 + s_3 < 1$, which means that we actually have $s_6 > 2$, or

$$c_1 + b_2 + a_3 \;>\; 2\,. \tag{1}$$

Similarly, since $s_1 + s_4 + s_5 = 3$, two of the numbers s_1, s_4, s_5 are less than $\frac{1}{2}$ (and the third is greater than 2).

- If $s_1 < \frac{1}{2}$ and $s_4 < \frac{1}{2}$, then $b_2 < \frac{1}{2}$ and $a_3 < \frac{1}{2}$, which implies $b_2 + a_3 < 1$. From (1) we have $c_1 > 1$, contradicting $c_1 \le c_1 + c_2 + c_3 = 1$.
- If $s_1 < \frac{1}{2}$ and $s_5 < \frac{1}{2}$, then $b_2 < \frac{1}{2}$ and $c_1 < \frac{1}{2}$, which implies $b_2 + c_1 < 1$. From (1) we have the contradiction $a_3 > 1$.
- If $s_4 < \frac{1}{2}$ and $s_5 < \frac{1}{2}$, then $a_3 < \frac{1}{2}$ and $c_1 < \frac{1}{2}$, which implies $a_3 + c_1 < 1$. From (1) we have the contradiction $b_2 > 1$.

In the same way it is readily checked that we are also led to a contradiction when the condition (1) is replaced by $s_2 > 2$ or $s_3 > 2$. The conclusion follows.

[*Editor's comment*: When this problem was originally printed, the interval that appeared was $\left[\frac{1}{3}, \frac{2}{3}\right]$. (This problem is impossible since the interval is too restricted.) A subsequent issue, [2001 : 213], (incorrectly) corrected the interval to $\left[\frac{1}{3}, \frac{3}{2}\right]$. This new problem can now be proven, but is not as sharp as it could be, since we can improve the lower bound on the interval. In the next issue, [2001 : 267], the problem was further corrected to the (correct) interval listed in the problem statement above. Unfortunately, some solvers missed this last correction, and solved the previously stated (weaker) problem. As a result, we have split the solvers into two groups: those who have solved the intended problem, and those who solved the weaker problem.]

Also solved by CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong; JOEL SCHLOSBERG, student, New York University, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. The weaker version of the problem was correctly solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ERIC POSTPISCHIL, Nashua, NH; and CHRIS WILDHAGEN, Rotterdam, the Netherlands. There was one incorrect solution.

Both Dimminie and the proposer show that these bounds are the best possible:

- $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $(0, 0, 1)$ *can give a sum of* $\frac{1}{2}$ *or* 2, *and*
- $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $(0, 0, 1)$, $(0, 0, 1)$ *can give a sum of* 0 *or* $\frac{3}{2}$.

The proposer also asks about generalizing the problem to n cards of n numbers each. He conjectures that the best intervals for achievable sums appear to be

$$\left[1 - \frac{2}{n}, 1 + \frac{2}{n}\right] \quad \text{for } n \text{ even}, \qquad \left[1 - \frac{2}{n+1}, 1 + \frac{2}{n+1}\right] \quad \text{for } n \text{ odd},$$

but he has no proof. Perhaps our readers can check these bounds and supply proofs!