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ON MOVING A SOFA AROUND A CORNER

ABSTRACT. A necessary condition is given for a region of the plane to have the greatest possible area of any region able to move around a right-angled corner in a hallway of unit width. A region is constructed, with area 2.2195... and bounded by 18 analytic pieces, which satisfies this condition. It is conjectured that this is the unique region of maximum area.

What is the greatest possible area for a sofa which can move around a right-angled corner in a hallway of unit width? We take the hallway H to be the set of points (x, y) such that x and y are both less than or equal to 1, and such that either x or y is greater than or equal to 0. The sofa S can be any connected region of the plane. S must be a subset of one of the two branches of H; without loss of generality, we require S to be a subset of the half strip $x \leqslant 1$, $0 \leqslant y \leqslant 1$. We must be able to rigidly move S, keeping it in H at all times, so that it ends up in the other branch of H, namely $y \leqslant 1$, $0 \leqslant x \leqslant 1$.

L. Moser [5] first published this question in 1966, although it seems to have had an earlier folk history. Hammersley [4, p. 84] conjectured that the answer is $\pi/2 + 2/\pi = 2.2074\ldots$, attained by a region composed of two quarter circles on either side of a 1 by $4/\pi$ rectangle from which a semicircle of radius $2/\pi$ has been removed (Figure 1). According to Conway [1], this problem and several related ones were discussed at a meeting in Copenhagen during the late 1960s, and a region (the 'Shephard piano') was constructed which slightly improved Hammersley's result, but this work was never published. Later, Conway and M. Guy proved that a sofa of maximum area (not necessarily unique) exists, while C. Francis and R. Guy [3] made several modifications of the Shephard piano to obtain an area of 2.215649. (The identification in [2] of the Shephard piano with Hammersley's solution seems to be wrong, since the

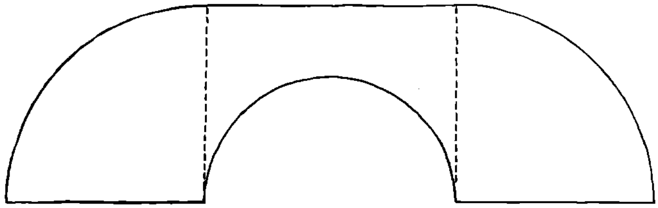

Fig. 1.

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procedure of Francis described in [3], if applied directly to Hammersley's solution, would yield an area of 2.2164.)

In this paper, Theorem 1 gives a strong condition which must hold for at least one region of maximal area, and in Theorem 2, a region (Figure 2) is constructed which satisfies this condition. This region has area 2.2195 . . . and is conjectured to be the unique region of maximal area, as well as the only region satisfying the condition which rotates through a right angle as it moves around the corner. It is of interest to note the close resemblance of Figure 2 to Wagner's [7] Monte Carlo solution. I have recently learned [6] that both theorems (or their equivalents, more or less) were conjectured in 1976 by B. Logan.

In the proofs of both theorems, we shall often consider the sofa to stay still while the hallway moves, so that S is the intersection of isometries of H.

DEFINITION. A polygon is *balanced* if, given any side, that side and every other side parallel to it lies on one of two lines, where the distance between the lines is 1 and the total length of the sides lying on each of the two lines is equal.

THEOREM 1. *There exists a real number* γ, $\pi/3 < \gamma \leqslant \pi/2$, *and a region* S, *such that* S *can move around the corner of* H, *rotating through an angle of* $-\gamma$ *in the process, such that no region of greater area can move around the corner, and such that for arbitrarily large* n, S *can be approximated arbitrarily closely by a polygonal region* P_n *with the following properties: The boundary of* P_n *is a balanced polygon.* P_n *is the intersection of* $n+1$ *sets* H_α *(where* $\alpha = k\gamma/n$ *and* $0 \leqslant k \leqslant n$*).* H_0 *is the half strip* $x \leqslant 1$, $0 \leqslant y \leqslant 1$. H_γ *is a translation of the half strip* $y \leqslant 1$, $0 \leqslant x \leqslant 1$ *rotated by angle* γ. *For* $0 < \alpha < \gamma$, H_α *is a translation of* H *rotated by angle* α.

Proof. The idea is to define a compact subset $\mathscr{T}$ of the set of all regions which can move around the corner, and show that $\mathscr{T}$ must include a region which has the maximum possible area of all such regions. We then construct

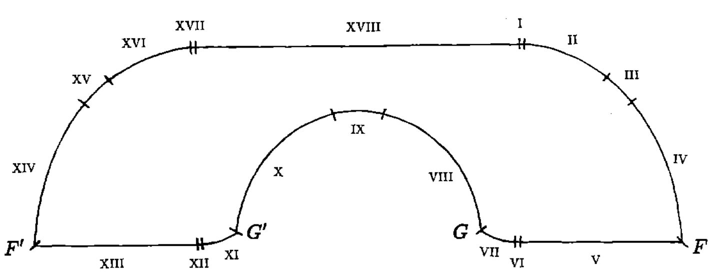

Fig. 2.

an infinite sequence of balanced polygonal regions $P_2, P_3, P_4, \dots$ and show that the area of each P_n is greater than or equal to the maximum area of any element of $\mathscr{T}$. Finally we construct an infinite sequence $T_2, T_3, T_4, \dots$ of elements of $\mathscr{T}$ such that T_n approximates P_n arbitrarily closely as n tends to infinity. It then follows from the compactness of $\mathscr{T}$ that the T_n, and hence the P_n, converge to some element of $\mathscr{T}$ which actually has the maximum possible area.

Given any real γ, let $T \in \mathscr{T} = \mathscr{T}_\gamma$ if and only if T is the intersection of a family of sets H_α, parametrized by all real α in the interval $0 \leqslant \alpha \leqslant \gamma$ (or, if $\gamma < 0, \gamma \leqslant \alpha \leqslant 0$), where the sets H_α satisfy the following conditions: H_0 is the half strip $x \leqslant 1$, $0 \leqslant y \leqslant 1$, H_γ is the half strip $y \cos\gamma - x \sin\gamma \leqslant 1 + q(\gamma)$, $p(\gamma) \leqslant x\cos\gamma + y\sin\gamma \leqslant 1 + p(\gamma)$, and H_α (for $0 < \alpha < \gamma$) is the set of all points (x, y) such that $x\cos\alpha + y \sin\alpha \leqslant 1 + p(\alpha)$, $y\cos\alpha - x \sin\alpha \leqslant 1 + q(\alpha)$, and either $x\cos\alpha + y\sin\alpha \geqslant p(\alpha)$ or $y\cos\alpha - x\sin\alpha \geqslant q(\alpha)$, where p and q are functions such that $p(0) = q(0) = 0$ and for each α in the interval $0 \leqslant \alpha \leqslant \gamma$ we have $x\cos\alpha + y\sin\alpha = 1 + p(\alpha)$ for some $(x, y) \in T$ and $y\cos\alpha - x\sin\alpha = 1 + q(\alpha)$ for some $(x, y) \in T$. If T meets these conditions, then we call $-\gamma$ an angle of rotation of T, and we call p and q the bounding functions of T. We must show firstly that if $T \in \mathscr{T}$ then T can move around the corner, rotating by angle $-\gamma$, and secondly that if a region S can move around the corner rotating by angle $-\gamma$, then there exists an element of $\mathscr{T}$ whose area is at least as great as the area of S.

Now if $T \in \mathscr{T}$, then T fits within the half strip $x \leqslant 1$, $0 \leqslant y \leqslant 1$, some translation of T rotated by $-\gamma$ fits within the half strip $y \leqslant 1$, $0 \leqslant x \leqslant 1$, and for every angle $-\alpha$ between 0 and $-\gamma$, some translation of T rotated by $-\alpha$ fits within H. In order to show that T can move around the corner of H, we need only show that the position of the isometry of T which fits into H at angle $-\alpha$ varies continuously with $-\alpha$, or, equivalently, that p and q are continuous functions.

In fact, we can prove more than continuity; we can prove that all p and q satisfy a uniform Lipschitz condition: for some $C > 0$, independent of γ, T, α, and β, if $|\alpha - \beta| < \delta$ then $|p(\alpha) - p(\beta)| < C\delta$ and $|q(\alpha) - q(\beta)| < C\delta$. It suffices to prove that all elements of $\mathscr{T}$ lie within a circle of radius C centered at the origin. For then, if $|\alpha - \beta| < \delta$ and $|p(\alpha) - p(\beta)| \geqslant C\delta$ (resp. $q(\alpha) - q(\beta)$), assuming without loss of generality that $p(\alpha) > p(\beta)$, we have $x\cos\alpha + y\sin\alpha = 1 + p(\alpha)$ for some $(x, y) \in T$, but for all such (x, y) within distance C of the origin, $x\cos\beta + y\sin\beta > 1 + p(\beta)$.

Since $p(0) = q(0) = 0$ by hypothesis, every element of $\mathscr{T}$ must include a point $(1, y)$ with $0 \leqslant y \leqslant 1$, so in order to prove that all elements of $\mathscr{T}$ lie within distance C of the origin, it suffices to demonstrate the existence of a uniform bound on the diameters of elements of $\mathscr{T}$. We consider three cases: $\gamma > \pi/4$, $-\pi/4 \leqslant \gamma \leqslant \pi/4$, and $\gamma < -\pi/4$. If $|\gamma| \leqslant \pi/4$, then H_0 and H_γ are subsets of two strips of width 1 whose edges intersect at an angle greater than or equal to $\pi/4$; hence the diameter of $H_0 \cap H_\gamma$, and the diameter of T, must be $\leqslant \sqrt{(1+\sqrt{2})^2 + 1^2}$. If $\gamma < -\pi/4$, then T is a subset of $H_0 \cap H_{-\pi/4}$, which again has a diameter $\leqslant \sqrt{(1+\sqrt{2})^2 + 1^2}$. If $\gamma > \pi/4$, then T is a subset of $H_0 \cap H_{\pi/4}$, which has a diameter $\leqslant 2 + 2\sqrt{2}$.

We have now proved that the diameters of all elements of $\mathscr{T}$ are uniformly bounded, from which it follows that all elements of $\mathscr{T}$ lie within a circle of radius C around the origin, and the functions p and q satisfy a uniform Lipschitz condition for all $T \in \mathscr{T}$, independent of γ. In particular, p and q are always continuous, so every element of $\mathscr{T}$ can move around the corner of H.

We must next show that no region outside of $\mathscr{T}$ which can move around the corner rotating by $-\gamma$ can have an area greater than that of every element of $\mathscr{T}$. In fact, we shall prove something stronger: No region outside of $\mathscr{T}$ which can fit into the first branch of H, fit into the second branch of H after rotation by $-\gamma$, and fit into H after rotation by every angle between 0 and $-\gamma$, can have an area greater than that of every element of $\mathscr{T}$. Indeed, let S be any region which can fit into the first and second branches of H after rotation by 0 and $-\gamma$ respectively, and can fit into H at every angle between 0 and $-\gamma$ (in particular, S can be any region which can move around the corner). We will construct a region T such that $T \in \mathscr{T}$ and some translation of S is a subset of T.

We assume, without loss of generality, that the maximum x and y coordinates of S are both 1 (if not, then use the appropriate translation of S). For $0 < \alpha \leqslant \gamma$ (or $\gamma \leqslant \alpha < 0$ if $\gamma < 0$), let $p(\alpha)$ be the maximum value of $x \cos\alpha + y \sin\alpha - 1$ and let $q(\alpha)$ be the maximum value of $y\cos\alpha - x\sin\alpha - 1$ for all $(x, y) \in S$. Let H_0, H_γ, and H_α (for α between 0 and γ) be defined as in the definition of $\mathscr{T}$, and let T be the intersection of H_0, H_γ, and all H_α. Then $T \in \mathscr{T}$ and, since for all α in the interval (including 0 and γ), some translation of S rotated by $-\alpha$ is in H, we have $S \subseteq H_\alpha$ for all α, so $S \subseteq T$.

We now prove that $\mathscr{T}$ is compact under the topology induced by the greater of the supnorms of p and q. Note that under this topology, the area of T is a continuous function of T, so the compactness of $\mathscr{T}$ implies that elements of $\mathscr{T}$ of maximum area exist.

Let $T_1, T_2, T_3, \ldots$ be an infinite sequence of elements of $\mathscr{T}$, and let p_i and q_i be the bounding functions of T_i. Let $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ be a countable set of points which is dense in the interval between 0 and γ. Both $|p_i(\alpha)|$ and $|q_i(\alpha)|$ are bounded by C, the maximum distance from the origin of any point of any element of $\mathscr{T}$. Therefore, for any fixed α, the sequence $(p_i(\alpha), q_i(\alpha))$ has an accumulation point. In particular, there exists a subsequence of $T_1, T_2, T_3, \ldots$ for which $(p_i(\alpha_1), q_i(\alpha_1))$ converges to some point $(p_\infty(\alpha_1), q_\infty(\alpha_1))$, this subsequence has a subsequence for which $(p_i(\alpha_2), q_i(\alpha_2))$ converges to some point $(p_\infty(\alpha_2), q_\infty(\alpha_2))$, and so on. By choosing one T_i with sufficiently large i from each of these subsequences, we can construct a sequence of T_i for which the points $(p_i(\alpha_j), q_i(\alpha_j))$ converge uniformly to points $(p_\infty(\alpha_j), q_\infty(\alpha_j))$ for all α_j. Since the α_j are dense in the interval between 0 and γ, and the functions p_i and q_i all satisfy a uniform Lipschitz condition, the sequences of functions p_i and q_i must converge uniformly to functions p_∞ and q_∞ over the closed interval from 0 to γ. The functions p_∞ and q_∞ can be used to construct a region T_∞. To prove that $\mathscr{T}$ is compact, we must show that $T_\infty \in \mathscr{T}$.

Suppose T_∞ is not in $\mathscr{T}$. Then for some α, either $x\cos\alpha + y\sin\alpha < 1 + p_\infty(\alpha)$ for all $(x, y) \in T_\infty$, or $y\cos\alpha - x\sin\alpha < 1 + q_\infty(\alpha)$ for all $(x, y) \in T_\infty$. But T_∞ is the intersection of closed sets H_α, and T_∞ is bounded by a circle of radius C, so T_∞ is compact and either $x\cos\alpha + y\sin\alpha - p_\infty(\alpha)$ or $y\cos\alpha - x\sin\alpha - q_\infty(\alpha)$ must have a maximum value $1 - \varepsilon$ with $\varepsilon > 0$. But this is impossible because T_∞ and its bounding functions can be approximated arbitrarily closely by regions T_i in $\mathscr{T}$ and their bounding functions. Hence $T_\infty \in \mathscr{T}$ and $\mathscr{T}$ is compact.

For each γ, therefore, the set $\mathscr{T}$ $(= \mathscr{T}_\gamma)$ has elements of maximum area. This maximum area must vary continuously with γ. We next show that the maximum area of any element of any of the sets $\mathscr{T}_\gamma$ must be attained by a region for which $\pi/3 < \gamma \leqslant \pi/2$. Now H_0 and H_γ are subsets of strips of unit width which intersect at angle $\pi/2 - \gamma$. If $|\gamma| \leqslant \pi/3$, then the intersection of these two strips is a rhombus of area $\leqslant 2$, so for such γ, the maximum area of any element of $\mathscr{T}_\gamma$ is $\leqslant 2$. But Hammersley's region is in $\mathscr{T}_{\pi/2}$ and has area $\pi/2 + 2/\pi > 2$. Suppose on the other hand that $\gamma < -\pi/3$. Then $T \in \mathscr{T}_\gamma$ would be a subset of $H_0 \cap H_{-\pi/4}$, which has area $\leqslant \sqrt{2}$. Suppose finally that $\gamma > \pi/2$. Let T^* be the intersection of all H_α for $0 \leqslant \alpha \leqslant \pi/2$, so that if $T \in \mathscr{T}_\gamma$ then $T \subseteq T^*$. Then T^* can be rotated $\pi/2$ radians within the hallway, starting from the first branch. But after this rotation, a simple translation to the right must suffice to move T^* into the second branch; otherwise it could not have fit into the first branch originally. Hence T^* can move around the corner, some element of $\mathscr{T}_{\pi/2}$ must have an area at least as great as that of T^*. Since the maximum area of any element of $\mathscr{T}_\gamma$ varies continuously with γ, is $\leqslant 2$ for $\gamma \leqslant \pi/2$, and never attains a greater value for $\gamma > \pi/2$ than for $\gamma = \pi/2$, the maximum value must exist and must be attained by some γ in the interval $\pi/3 < \gamma \leqslant \pi/2$.

We now choose some γ in the interval $\pi/3 < \gamma \leqslant \pi/2$ such that one of the regions in $\mathscr{T}$ $(= \mathscr{T}_\gamma)$ has the maximum possible area of any region which can move around the corner. For each integer $n \geqslant 2$, let P_n be a region of maximum possible area which fits into the first branch of H, which fits into the second branch of H after rotation by angle $-\gamma$, and which fits somewhere in H after rotation by $-k\gamma/n$ for all integers k, $1 \leqslant k \leqslant n-1$. It is clear that P_n exists, because any such region must be the intersection of H_0, H_γ, and $H_{k\gamma/n}$ for $1 \leqslant k \leqslant n-1$, where these sets are defined as before. The same argument as in the continuous case can be used to show that the diameter of such a region, and hence $|p(k\gamma/n)|$ and $|q(k\gamma/n)|$, are bounded (the bound C might be greater in the discrete case, because $k\gamma/n$ is not necessarily equal to $\pi/4$ for any k, but since $\pi/3 < \gamma \leqslant \pi/2$ and $n \geqslant 2$, we must have $\pi/6 \leqslant k\gamma/n \leqslant \pi/3$ for some k, so the diameter of $H_0 \cap H_{k\gamma/n}$ is $\leqslant 2\sqrt{3}+2$). Since the area of the region varies continuously with $p(k\gamma/n)$ and $q(k\gamma/n)$ for $1 \leqslant k \leqslant n$, and since $[-C, C]^n$ is compact, the maximum possible area must be attained by some region P_n.

Now P_n must be bounded by a polygon, since it is the intersection of a finite number of regions $H_{k\gamma/n}$ bounded by straight lines. Each side of this polygon lies on one of the lines

$$x \cos k\gamma/n + y \sin k\gamma/n = p(k\gamma/n),$$

$$x \cos k\gamma/n + y \sin k\gamma/n = 1 + p(k\gamma/n),$$

$$y \cos k\gamma/n - x \sin k\gamma/n = q(k\gamma/n),$$

or

$$y \cos k\gamma/n - x \sin k\gamma/n = 1 + q(k\gamma/n)$$

for $0 \leqslant k \leqslant n$ (the first and third equations are excluded when $k = 0$). If two sides are parallel, then the equations of their lines must have the same k. This is obvious in the case where $\gamma < \pi/2$. In the case where $\gamma = \pi/2$, we have

$$x \cos 0 + y \sin 0 = 1 + p(0)$$

parallel to $y \cos \pi/2 - x \sin \pi/2 = 1 + q(\pi/2)$, but in fact neither of these vertical lines can include sides of P_n. If P_n had vertical sides, then we could increase the area of P_n by either decreasing all $p(k\gamma/n)$ except $p(0)$ by the same amount (which would have the effect of moving the half strip H_0 to the right relative to all the other $H_{k\gamma/n}$) or by increasing $q(\gamma)$ (which would move the half strip H_γ to the left). We also have

$$y \cos 0 - x \sin 0 = q(0) \quad \text{and} \quad y \cos 0 - x \sin 0 = 1 + q(0)$$

parallel to

$$x \cos \pi/2 + y \sin \pi/2 = p(\pi/2)$$

and

$$x \cos \pi/2 + y \sin \pi/2 = 1 + p(\pi/2),$$

but the two sets of lines are in fact identical, because if $p(\pi/2) \neq q(0) = 0$, then we could increase the area of P_n by increasing or decreasing $p(\pi/2)$.

Now suppose the boundary of P_n were not a balanced polygon. Then we would have, for some k, the length of the sides on the line $x \cos k\gamma/n + y \sin k\gamma/n = p(k\gamma/n)$ less than or greater than the length of the sides on $x \cos k\gamma/n + y \sin k\gamma/n = 1 + p(k\gamma/n)$ or the length of the sides on $y \cos k\gamma/n - x \sin k\gamma/n = q(k\gamma/n)$ less than or greater than the length of the sides on $y \cos k\gamma/n - x \sin k\gamma/n = 1 + q(k\gamma/n)$. Suppose the length of the sides on $x \cos k\gamma/n + y \sin k\gamma/n = p(k\gamma/n)$ is less than the length of the sides on $x \cos k\gamma/n + y \sin k\gamma/n = 1 + p(k\gamma/n)$. Then we would increase the area of P_n by increasing $p(k\gamma/n)$. In the other three cases, we could achieve the same result by decreasing $p(k\gamma/n)$, increasing $q(k\gamma/n)$, or decreasing $q(k\gamma/n)$ respectively. (When $k = 0$, we have $q(0) = 0$ by definition, so instead of increasing $q(0)$ we must decrease all the other $q(k\gamma/n)$, or vice versa.) Therefore the regions P_n are all bounded by balanced polygons.

Suppose $T \in \mathscr{T}$. Then T is the intersection of regions H_α for $0 \leqslant \alpha \leqslant \gamma$, and is therefore, for each $n \geqslant 2$, a subset of the intersection of the $n+1$ regions $H_{k\gamma/n}$ for $0 \leqslant k \leqslant n$. Since the area of the latter intersection is less than or equal to the area of P_n, it follows that the area of every element of $\mathscr{T}$ is less than or equal to the area of every P_n.

For each $n \geqslant 2$, we now construct a region $T_n \in \mathscr{T}$ by letting $p(\alpha)$ and $q(\alpha)$ be the maximum values of $x \cos \alpha + y \sin \alpha - 1$ and $y \cos \alpha - x \sin \alpha - 1$ respectively for $(x, y) \in P_n$. Then for sufficiently large n, P_n is approximated arbitrarily closely by T_n in the sense that points in $P_n - T_n$ are arbitrarily close to T_n. Since $\mathscr{T}$ is compact, the sequence T_2, T_3, $T_4, \ldots$ must have an accumulation point $T_\infty \in \mathscr{T}$. For arbitrarily large n, T_∞ can be approximated arbitrarily closely by T_n, and therefore by P_n as well. Since the P_n have areas greater than or equal to that of every element of $\mathscr{T}$, the area of T_∞ must be the maximum in $\mathscr{T}$. This proves Theorem 1.

THEOREM 2. *There exist positive real numbers φ, θ, A, and B (indeed, to an accuracy of five decimal places, $\varphi = 0.03918$, $\theta = 0.68130$, $A = 0.09443$, and*

$B = 1.39920$) *such that the following holds*: *Let*

$$r(\alpha) = \begin{cases} \dfrac{1}{2} & \text{for } 0 < \alpha < \varphi \\ \dfrac{1}{2} + \dfrac{1}{2}A + \dfrac{1}{2}\alpha - \dfrac{1}{2}\varphi & \text{for } \varphi < \alpha < \theta \\ A + \alpha - \varphi & \text{for } \theta < \alpha < \dfrac{\pi}{2} - \theta \\ B - \dfrac{1}{2}\left(\dfrac{\pi}{2} - \alpha - \varphi\right)(1 + A) - \dfrac{1}{4}\left(\dfrac{\pi}{2} - \alpha - \varphi\right)^2 & \text{for } \dfrac{\pi}{2} - \theta < \alpha < \dfrac{\pi}{2} - \varphi. \end{cases}$$

For $0 \leqslant \alpha \leqslant \pi/2 - \varphi$, *let*

$$y(\alpha) = \int_\alpha^{\pi/2 - \varphi} r(t) \sin t \, dt$$

and let

$$x(\alpha) = 1 - \int_\alpha^{\pi/2 - \varphi} r(t) \cos t \, dt.$$

Let

$$p(\alpha) = \begin{cases} \cos \alpha - 1 & \text{for } 0 \leqslant \alpha \leqslant \varphi \\ x\left(\dfrac{\pi}{2} - \alpha\right) \cos \alpha + y\left(\dfrac{\pi}{2} - \alpha\right) \sin \alpha - 1 & \text{for } \varphi \leqslant \alpha \leqslant \dfrac{\pi}{2} \end{cases}$$

and

$$q(\alpha) = \begin{cases} y(\alpha) \cos(\alpha) - [4x(0) - 2 - x(\alpha)] \sin \alpha - 1 & \text{for } 0 \leqslant \alpha \leqslant \dfrac{\pi}{2} - \varphi \\ -[4x(0) - 3] \sin \alpha - 1 & \text{for } \dfrac{\pi}{2} - \varphi \leqslant \alpha \leqslant \dfrac{\pi}{2}. \end{cases}$$

Let $\gamma = \pi/2$ *and let* H_0, H_γ, *and* H_α $(0 < \alpha < \gamma)$ *be defined as in the proof of Theorem* 1. *Let* $S = \bigcap_{0 \leqslant \alpha \leqslant \pi/2} H_\alpha$. *Then for arbitrarily large n, S can be approximated arbitrarily closely by a region* P_n *having the properties of* P_n *in Theorem* 1.

Proof. We first give some conditions on a polygonal region P_n and show that, for sufficiently large n, these conditions imply that P_n has the properties of P_n in Theorem 1. We then show that, for sufficiently large n, polygonal regions exist which satisfy these conditions, and that these regions approximate S arbitrarily closely.

We assume that $P_n = \bigcap_{k=0}^{n} H_{k\delta}$, where $\delta = \pi/2n$, $H_0 = \{(x, y) \mid x \leqslant 1$ and $0 \leqslant y \leqslant 1\}$, $H_{\pi/2} = \{(x, y) \mid x \geqslant -1 - q_n(\pi/2)$ and $0 \leqslant y \leqslant 1\}$ and $H_{k\delta} = \{(x, y) \mid x \cos k\delta + y \sin k\delta \leqslant 1 + p_n(k\delta),\ y \cos k\delta - x \sin k\delta \leqslant 1 + q_n(k\delta)$, and either $x \cos k\delta + y \sin k\delta \geqslant p_n(k\delta)$ or $y \cos k\delta - x \sin k\delta \geqslant q_n(k\delta)\}$ for $1 \leqslant k \leqslant n-1$, where p_n and q_n are real-valued functions on the set of points $k\delta, 0 \leqslant k \leqslant n$. In order to show that P_n has the properties listed in Theorem 1, we need only show that the following conditions imply that P_n is a balanced polygon.

The first condition is that P_n has a vertical axis of symmetry, or equivalently, that $q_n(\pi/2 - k\delta) = p_n(k\delta) - q_n(\pi/2)\cos k\delta$ and $p_n(\pi/2 - k\delta) = q_n(k\delta) - q_n(\pi/2)\sin k\delta$ for all k $(0 \leqslant k \leqslant n)$. We also require that for all k, $y \cos k\delta - x \sin k\delta = 1 + q_n(k\delta)$ for some $(x, y) \in P_n$ (where $q_n(0) = 1$). Before stating the rest of the conditions, we require some definitions.

For $0 \leqslant k \leqslant n-1$, let ρ_k be the point of intersection of the lines

$$y \cos k\delta - x \sin k\delta = 1 + q_n(k\delta)$$

and

$$y \cos(k+1)\delta - x \sin(k+1)\delta = 1 + q_n((k+1)\delta),$$

let σ_k be the point of intersection of the lines

$$y \cos k\delta - x \sin k\delta = q_n(k\delta)$$

and

$$y \cos(k+1)\delta - x \sin(k+1)\delta = q((k+1)\delta),$$

and let ζ_k be the point of intersection of the lines

$$y \cos k\delta - x \sin k\delta = q_n(k\delta)$$

and

$$x \cos(k+1)\delta + y \sin(k+1)\delta = p_n((k+1)\delta).$$

For $0 \leqslant k \leqslant n$, let ω_k be the point of intersection of the lines

$$x \cos k\delta + y \sin k\delta = p_n(k\delta) \quad \text{and} \quad y \cos k\delta - x \sin k\delta = q_n(k\delta).$$

Note that ζ_{n-k-1} and ω_{n-k} are the reflections of ζ_k and ω_k respectively about the vertical axis of symmetry. Let ρ_{-1} be the point $\tan\frac{1}{2}\delta$ units to the right of ρ_0 and let σ_{-1} be the point $\tan\frac{1}{2}\delta$ units to the left of σ_0.

For $0 \leqslant k \leqslant n-1$, let r_k be the length of segment $\rho_{k-1}\rho_k$, let u_k be the length of segment $\sigma_{k-1}\omega_k$, and let s_k be u_k minus the length of $\sigma_k\omega_k$ (that is, s_k is plus or minus the length of segment $\sigma_{k-1}\sigma_k$, depending on whether σ_k or

σ_{k-1} is closer to ω_k). Note that segment $\zeta_{k-1}\omega_k$ has length $u_k \tan\delta$ (for $1 \leqslant k \leqslant n$) and, because of the symmetry condition, segment $\omega_k\zeta_k$ has length $u_{n-k} \tan\delta$ (for $0 \leqslant k \leqslant n-1$).

The requirement that the outer boundary of each isometry of H must actually touch P_n implies that for each k ($1 \leqslant k \leqslant n-1$), the segment $\rho_{k-1}\rho_k$ is a side of P_n, unless $\rho_{k-1} = \rho_k$, in which case this point is a vertex of P_n.

For $1 \leqslant k \leqslant n-1$, let a_k be the fraction of segment $\sigma_{k-1}\sigma_k$, and b_k the fraction of segment $\zeta_k\omega_k$ (or equivalently, $\zeta_{n-k-1}\omega_{n-k}$) which is on the boundary of P_n. The following additional conditions then imply that the boundary of P_n is a balanced polygon:

$$(1) \qquad r_k = a_k s_k + b_k u_{n-k} \tan\delta \quad \text{for } 1 \leqslant k \leqslant n-1,$$

$$(2) \qquad u_0 + 1 - 2\tan\frac{1}{2}\delta = 3\sum_{k=1}^{n-1} r_k \cos k\delta,$$

$$(3) \qquad \sum_{k=1}^{n-1} r_k \sin k\delta = 1.$$

Condition (1) states that r_k, the length of the side of P_n on the line $y\cos k\delta - x\sin k\delta = 1 + q_n(k\delta)$, is equal to the length of the side, or sides, on the parallel line $y\cos k\delta - x\sin k\delta = q_n(k\delta)$. The symmetry condition then implies that the same is true of the sides on the lines $x\cos k\delta + y\sin k\delta = 1 + p_n(k\delta)$ and $x\cos k\delta + y\sin k\delta = p_n(k\delta)$. Condition (2) implies that the length of the side of P_n on the line $y=1$, namely $u_0 + 1 - \Sigma_{k=1}^{n-1} r_k \cos k\delta$, is twice the length of each of the two sides on the x-axis, namely $\Sigma_{k=1}^{n-1} r_k \cos k\delta + \tan\frac{1}{2}\delta$. Condition (3) implies that ρ_{n-1} is on the x-axis, so P_n has no vertical sides. It follows that P_n is balanced.

We must now show that for all sufficiently large n, there exist functions p_n and q_n such that, if $\lim_{n\to\infty} \alpha_n = \alpha$, then $\lim_{n\to\infty} p_n(\alpha_n) = p(\alpha)$ and $\lim_{n\to\infty} q_n(\alpha_n) = q(\alpha)$. We will do this by first solving for r_k, s_k, and u_k ($1 \leqslant k \leqslant n-1$) in terms of u_0 and two other parameters, φ and θ, and then solving for u_0, φ, and θ.

We define φ and θ as follows: Let E be the union of all segments $\sigma_{k-1}\sigma_k$ for $1 \leqslant k \leqslant n$ and let W be the union of all segments $\omega_{k-1}\omega_k$ for $1 \leqslant k \leqslant n$. Let $\theta^*: E \to [0, \pi/2]$ and $\phi^*: W \to [0, \pi/2]$ be the functions for which $\theta^*(\sigma_k) = k\delta$ and $\varphi^*(\omega_k) = k\delta$ for $0 \leqslant k \leqslant n$ and such that both θ^* and φ^* are linear on each segment of their domains. In the case where E and W intersect at exactly one point, we let θ and φ be the values of θ^* and φ^* respectively at that point. Let μ and ν be the least integers $\geqslant \varphi/\delta$ and θ/δ respectively.

We now consider the appearance of E and W near their point of intersection, for a certain range of values of ϕ, θ, u_μ and $u_{n-\mu}$ when n is large.

The angle $k\delta$ of the segment $\sigma_{k-1}\sigma_k$ changes slowly with k, so E can be approximated by a line making an angle of θ with the x-axis. Likewise, each segment $\zeta_{n-k}\omega_{n-k}$, for k near μ, makes an angle of approximately φ with the y-axis, while $\omega_{n-k+1}\zeta_{n-k}$ is nearly perpendicular to $\zeta_{n-k}\omega_{n-k}$, so $\omega_{n-k+1}\omega_{n-k}$ makes an angle of approximately $\arctan u_{n-\mu}/u_\mu$ with $\zeta_{n-k}\omega_{n-k}$, and W can be approximated by a line which makes an angle of $\varphi+\arctan u_{n-\mu}/u_\mu$ with the y-axis. Suppose that, for all sufficiently large n, $\theta+\varphi+\arctan u_{n-\mu}/u_\mu$ is bounded from above by some number strictly less than $\pi/2$ and independent of n. Then W has a greater slope than E near their point of intersection. It follows that for all $k < \nu$, the segment $\sigma_{k-1}\sigma_k$ is one of the sides of P_n, and $a_k = 1$. For $k = \nu$, and for a small number of k immediately after ν (where the small number does not depend on n), some part of the segment $\sigma_{k-1}\sigma_k$ might be one of the sides of P_n (so $0 < a_k < 1$), but for all larger k, the segment $\sigma_{k-1}\sigma_k$ lies entirely outside of P_n, and $a_k = 0$. Likewise, for $k < \mu$, the segments $\zeta_{n-k}\omega_{n-k}$ and $\omega_{n-k}\zeta_{n-k-1}$ are entirely outside P_n and, by the symmetry condition, the same is true for $k > n-\mu$, so for k in these ranges, $b_k = 0$. For $k = \mu$, and a small number of k immediately after μ (as well as $k = n-\mu$, and a small number of k immediately before $n-\mu$), some parts of $\zeta_{n-k}\omega_{n-k}$ and $\omega_{n-k}\zeta_{n-k-1}$ might be on the boundary of P_n (so $0 < b_k < 1$), while for all other k, from shortly after μ to shortly before $n-\mu$, the entire segments $\zeta_{n-k}\omega_{n-k}$ and $\omega_{n-k}\zeta_{n-k-1}$ are sides of P_n, and $b_k = 1$ (Figure 3).

Now suppose we are given φ, θ, and u_0, with $\varphi < \theta$, so that $\mu < \nu$, and

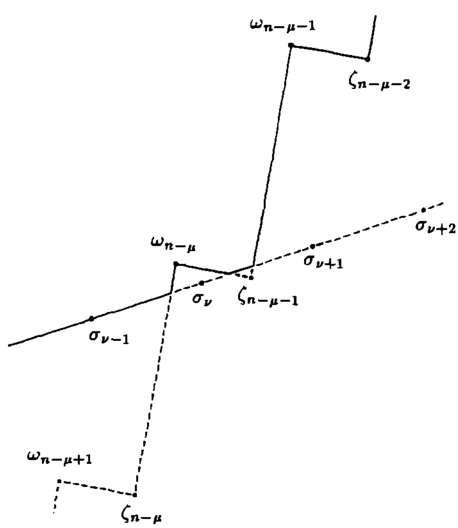

Fig. 3.

suppose n is large enough that there is no overlap between the values of k for which $0 < a_k < 1$ and the values of k for which $0 < b_k < 1$. For $n-\mu+1 \leqslant k \leqslant n-1$, we have $a_k = b_k = 0$, so by (1), $r_k = 0$. Since segments $\rho_{k-2}\rho_{k-1}$, $\rho_{k-1}\rho_k$, and $\rho_k\rho_{k+1}$ are parallel to, and distance 1 from segments $\sigma_{k-2}\sigma_{k-1}$, $\sigma_{k-1}\sigma_k$, and $\sigma_k\sigma_{k+1}$ respectively, it follows that

(4) $$r_k + s_k = 2\tan\tfrac{1}{2}\delta \quad \text{for } 1 \leqslant k \leqslant n-1.$$

Therefore, for $n-\mu+1 \leqslant k \leqslant n-1$, we have $s_k = 2\tan\frac{1}{2}\delta$. Likewise, for $1 \leqslant k \leqslant \mu-1$, we have $a_k = 1$ and $b_k = 0$, so $r_k = s_k = \tan\frac{1}{2}\delta$.

By the definitions of u_k and s_k, and the symmetry condition, we have

(5) $$u_k = u_{k+1}\sec\delta + s_k + u_{n-k}\tan\delta \quad \text{for } 0 \leqslant k \leqslant n-1.$$

Also, since ρ_{n-1} is on the x-axis, the x-coordinate of σ_{n-1} is $-\tan\frac{1}{2}\delta$, so

(6) $$u_n = \tan\tfrac{1}{2}\delta.$$

Starting with u_0 and u_n, we can repeatedly use (5) to find u_k for $1 \leqslant k \leqslant \mu$ and $n-\mu+1 \leqslant k \leqslant n-1$.

We can proceed further only if we know b_μ, $b_{n-\mu}$ and the few adjacent values of b_k which can be less than 1. It turns out in fact that b_k can lie strictly between 0 and 1 only if $k = \mu$, $n-\mu$, or $n-\mu-1$, and we will use this set of values for the sake of concreteness, but the same argument holds for any set of values, provided the number of values of k remains fixed as $n \to \infty$. We can find b_μ, $b_{n-\mu}$, and $b_{n-\mu-1}$ by geometric construction if we know φ, θ, u_μ, $u_{n-\mu}$, $u_{\mu+1}$, s_ν, $s_{\nu+1}$, and $s_{\nu+2}$. (Again, it turns out that a_ν, $a_{\nu+1}$, and $a_{\nu+2}$ are the only a_k which lie strictly between 0 and 1, but we need not prove this; our argument will work with any fixed number of values of k.) We already know φ, θ, and u_μ, and if we assume values for s_ν, $s_{\nu+1}$, and $s_{\nu+2}$, then we can solve for $b_{n-\mu}$, and then solve for $u_{n-\mu}$, b_μ, $u_{\mu+1}$, and $b_{n-\mu-1}$. For $\mu+1 \leqslant k \leqslant n-\mu-2$, we have $b_k = 1$, and for $k \leqslant \nu-1$, we have $a_k = 1$, while $a_k = 0$ for $k \geqslant \nu+3$, so we can find r_k, s_k, and u_k for $\mu+1 \leqslant k \leqslant \nu-1$ and $n-\nu+1 \leqslant k \leqslant n-\mu-1$. Our trial values for s_ν, $s_{\nu+1}$, and $s_{\nu+2}$ then give us a_ν, $a_{\nu+1}$, and $a_{\nu+2}$, which allow us to derive corrected values for s_ν, $s_{\nu+1}$, and $s_{\nu+2}$. Since $\sigma_{\nu-1}$, σ_ν, $\sigma_{\nu+1}$, and $\sigma_{\nu+2}$ are very nearly collinear, an error of ε in s_ν, $s_{\nu+1}$, and $s_{\nu+2}$ will result in an error of $O(\delta\varepsilon)$ in $b_{n-\mu}$, $u_{n-\mu}$, b_μ, $u_{\mu+1}$, and $b_{n-\mu-1}$. For higher values of k, b_k is exactly 1, so each successive value of u_k accumulates an additional error of only $O(\delta^2\varepsilon)$. The total error in the corrected values of s_ν, $s_{\nu+1}$, and $s_{\nu+2}$ will therefore be $O(\delta\varepsilon)$. Repeating the calculations with the corrected values will cause s_ν, $s_{\nu+1}$, and $s_{\nu+2}$ to quickly converge to their true values, and we simultaneously obtain true values for r_k, s_k, and u_k for all k between μ and ν and between $n-\nu$ and $n-\mu$.

Finally, for $\nu+3 \leqslant k \leqslant n-\mu-2$, we have $a_k=0$ and $b_k=1$ so

$$u_k = u_{k+1} \sec\delta + 2\tan\tfrac{1}{2}\delta \tag{7}$$

and

$$u_k = u_{n-\mu-1}(\sec\delta)^{n-\mu-1-k} + \frac{(\sec\delta)^{n-\mu-1-k}-1}{\sec\delta-1}. \tag{8}$$

If we plug $k=\nu+3$ into this last formula, we will not necessarily get a value for $u_{\nu+3}$ which satisfies $u_{\nu+2}=u_{\nu+3}\sec\delta+s_{\nu+2}+u_{n-\nu-2}\tan\delta$ unless we choose the correct values for φ, θ, and u_0. But for any choice of φ, θ, and u_0, we get some values for

$$u_{\nu+3}\sec\delta+s_{\nu+2}+u_{n-\nu-2}\tan\delta-u_{\nu+2} \tag{9}$$

$$\sum_{k=1}^{n} r_k \sin k\delta - 1 \tag{10}$$

and

$$u_0+1-2\tan\frac{1}{2}\delta-3\sum_{k=1}^{n} r_k\cos k\delta \tag{11}$$

which vary continuously with φ, θ, and u_0. If all three of the above expressions are zero, then P_n is balanced.

As $n\to\infty$, the expressions (9), (10), and (11), as functions of φ, θ, and u_0, approach three functions defined as follows: Let $r(\alpha)$, $s(\alpha)$, and $u(\alpha)$ be functions of α, for $0<\alpha<\pi/2$, such that

$$r(\alpha)+s(\alpha)=1 \tag{12}$$

$$u'(\alpha) = -s(\alpha)-u\left(\frac{\pi}{2}-\alpha\right) \quad \text{for } \alpha\neq\varphi,\ \theta,\ \frac{\pi}{2}-\varphi \tag{13}$$

$$u\left(\frac{\pi}{2}\right)=0 \tag{14}$$

$$u(0)=u_0 \tag{15}$$

and

$$r(\alpha)=\begin{cases} s(\alpha) & \text{if } 0<\alpha<\varphi \\ s(\alpha)+u\left(\frac{\pi}{2}-\alpha\right) & \text{if } \varphi<\alpha<\theta \\ u\left(\frac{\pi}{2}-\alpha\right) & \text{if } \theta<\alpha<\frac{\pi}{2}-\varphi \\ 0 & \text{if } \frac{\pi}{2}-\varphi<\alpha<\frac{\pi}{2}. \end{cases} \tag{16}$$

Expressions (9), (10), and (11) then approach, respectively,

(17) $$\lim_{\alpha\downarrow\theta} u(\alpha) - \lim_{\alpha\uparrow\theta} u(\alpha)$$

(18) $$\int_0^{\pi/2} r(\alpha)\sin\alpha\, d\alpha - 1$$

and

(19) $$u(0) + 1 - 3\int_0^{\pi/2} r(\alpha)\cos\alpha\, d\alpha.$$

Equations (12) through (16) are satisfied if and only if

(20) $$r(\alpha) = \begin{cases} \frac{1}{2} & \text{for } 0 < \alpha < \varphi \\ \frac{1}{2}(1 + A + \alpha - \varphi) & \text{for } \varphi < \alpha < \theta \\ A + \alpha - \varphi & \text{for } \theta < \alpha < \frac{\pi}{2} - \theta \\ B - \frac{1}{2}\left(\frac{\pi}{2} - \alpha - \varphi\right)(1 + A) - \frac{1}{4}\left(\frac{\pi}{2} - \alpha - \varphi\right)^2 & \text{for } \frac{\pi}{2} - \theta < \alpha < \frac{\pi}{2} - \varphi \\ 0 & \text{for } \frac{\pi}{2} - \varphi < \alpha < \frac{\pi}{2} \end{cases}$$

(21) $$s(\alpha) = \begin{cases} \frac{1}{2} & \text{for } 0 < \alpha < \varphi \\ \frac{1}{2}(1 - A - \alpha + \varphi) & \text{for } \varphi < \alpha < \theta \\ 1 - A - \alpha + \varphi & \text{for } \theta < \alpha < \frac{\pi}{2} - \theta \\ 1 - B + \frac{1}{2}\left(\frac{\pi}{2} - \alpha - \varphi\right)(1 + A) + \frac{1}{4}\left(\frac{\pi}{2} - \alpha - \varphi\right)^2 & \text{for } \frac{\pi}{2} - \theta < \alpha < \frac{\pi}{2} - \varphi \\ 1 & \text{for } \frac{\pi}{2} - \varphi < \alpha < \frac{\pi}{2} \end{cases}$$

(22) $$u(\alpha) = \begin{cases} B - \frac{1}{2}(\alpha - \varphi)(1 + A) - \frac{1}{4}(\alpha - \varphi)^2 & \text{for } \varphi < \alpha < \theta \\ A + \frac{\pi}{2} - \varphi - \alpha & \text{for } \theta < \alpha < \frac{\pi}{2} - \varphi \end{cases}$$

$$(23)\qquad A\cos\varphi=\sin\varphi+\tfrac{1}{2}-\tfrac{1}{2}\cos\varphi+B\sin\varphi$$

and

$$(24)\qquad u_0+1=\cos\varphi+A\sin\varphi+B\cos\varphi+\tfrac{1}{2}\sin\varphi,$$

where $A=u(\pi/2-\varphi)$ and $B=u(\varphi)$. The cases $0<\alpha<\varphi$ and $\pi/2-\varphi<\alpha<\pi/2$ follow immediately from (12) and (16). From (12), (13) and (16) we obtain $u'(\alpha)=-1$ for $\theta<\alpha<\pi/2-\varphi$. We can then integrate $u'(\alpha)$ to obtain $u(\alpha)$ over this range, and apply (12) and (16) to obtain $s(\alpha)$ and $r(\alpha)$ for $\varphi<\alpha<\pi/2-\theta$. We can now use (13) to find $u'(\alpha)$ over the range $\varphi<\alpha<\theta$, and integrate once more to obtain the remaining values of $u(\alpha)$, $s(\alpha)$, and $r(\alpha)$. To derive (23) and (24), we first substitute the right-hand side of (21) for s in (13) in the cases $\pi/2-\varphi<\alpha<\pi/2$ and $0<\alpha<\varphi$ respectively. We then differentiate both sides of (13) and substitute the right-hand side of (13) for u' to obtain $u''(\alpha)+u'(\alpha)+\frac{1}{2}=0$ for $\pi/2-\varphi<\alpha<\pi/2$ and $u''(\alpha)+u'(\alpha)+1=0$ for $0<\alpha<\varphi$. It follows that for $\pi/2-\varphi<\alpha<\pi/2$ (resp. $0<\alpha<\varphi$) $u(\alpha)+\frac{1}{2}$ (resp. $u(\alpha)+1$) is a linear combination of $\sin\alpha$ and $\cos\alpha$. By hypothesis $u(\pi/2-\varphi)=A$ and $u(\varphi)=B$, and from (13) and (21) we know that the right derivative of u at $\pi/2-\varphi$ is $-1-B$ and the left derivative at φ is $-\frac{1}{2}-A$, so we can solve for u in both ranges. We then obtain (23) and (24) from (14) and (15) respectively, and from the continuity of u.

The expressions (17), (18), and (19) are all equal to zero if, in addition to (20), (21), and (22), we have

$$(25)\qquad A+\frac{\pi}{2}-\varphi-\theta=B-\frac{1}{2}(\theta-\varphi)(1+A)+\frac{1}{4}(\theta-\varphi)^2$$

$$(26)\qquad B\sin\varphi=\int_{\varphi}^{\theta}s(\alpha)\sin\alpha\,d\alpha$$

and

$$(27)\qquad 2\left(\cos\varphi+A\sin\varphi+B\cos\varphi+\frac{1}{2}\sin\varphi\right)=3\left(\sin\varphi+B\cos\varphi+\int_{\varphi}^{\theta}s(\alpha)\cos\alpha\,d\alpha\right).$$

The above equations can be obtained directly if we use (20), (22), and (24) to replace $r(\alpha)$, $u(\alpha)$, and $u(0)$ in (17), (18), and (19).

We now use Equations (23), (25), (26), and (27) to solve for A, B, φ, and θ. Equation (26), after substituting for s from (21) and integrating, gives us B in terms of A, φ, and θ, and (23) can then be used to eliminate A. We are left with two equations in φ and θ, which can be solved numerically to yield approximate values of 0.03918 for φ and 0.68130 for θ. By computing the partial derivatives of both equations with respect to φ and θ, we can show that these solutions are unique within some neighborhood. Substituting back, we get values of approximately 0.09443 for A and 1.39920 for B, and (24) then gives us 1.42064 for u_0.

It follows that, for sufficiently large n, there must be values of φ, θ, and u_0 arbitrarily close to 0.03918…, 0.68130…, and 1.42064… respectively, for which the expressions (9), (10), and (11) are all zero. For these values, P_n approaches S arbitrarily closely. This completes the proof of Theorem 2.

REMARK. S, our conjectured sofa of maximum area, is shown in Figure 2. The boundary of S consists of 18 analytic pieces: V, XIII, and XVIII are line segments, I, VI, XII, and XVII are circular arcs, II, III, VII, XI, XV, and XVI are involutes of circles, and IV and XIV are involutes of involutes of circles. The entire figure has a vertical axis of symmetry. The lines tangent to pieces I, II, III, and IV have angle $-\alpha$ with respect to the x-axis, where α ranges from 0 to φ, φ to θ, θ to $\pi/2-\theta$, and $\pi/2-\theta$ to $\pi/2-\varphi$ respectively. The radius of curvature of each of these pieces at the point of tangency is $r(\alpha)$. For pieces VI and VII, α ranges from 0 to φ and φ to θ respectively, and the radius of curvature is $s(\alpha)$. Let $-\alpha$ be the angle of rotation of S from its initial position as it travels around the corner of the hallway H. When $\alpha=0$, piece XVIII is on the line $y=1$, XIII is on the x-axis, and point F is on the line $x=1$. For $0<\alpha<\varphi$, F remains on $x=1$, while $y=1$ and the x-axis are tangent to XVII and XII respectively. At $\alpha=\varphi$, point G coincides with the origin, $x=1$ is tangent to IV at F, and the points where $y=1$ and the x-axis are tangent to the curve move from XVII to XVI and from XII to XI respectively. When $\varphi<\alpha<\theta$, VIII passes through the origin, $x=1$ is tangent to IV, $y=1$ is tangent to XVI, and the x-axis is tangent to XI. When $\alpha=\theta$, the x-axis is tangent to XI at G', and when $\theta<\alpha<\pi/2-\theta$, the boundary of the isometry of S touches the boundary of H at only three points: IX passes through the origin, $x=1$ is tangent to III and $y=1$ is tangent to XV. Indeed, at $\alpha=\pi/4$, G misses the y-axis and G' misses the x-axis by only 0.0012.

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