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On topological set theory

Thierry Libert* and **Olivier Esser****

Département de Mathématique, Université Libre de Bruxelles,
Boulevard du Triomphe, CP211, 1050 Bruxelles, Belgium

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This paper is concerned with topological set theory, and particularly with Skala's and Manakos' systems for which we give a topological characterization of the models. This enables us to answer natural questions about those theories, reviewing previous results and proving new ones. One of these shows that Skala's set theory is in a sense compatible with any 'normal' set theory, and another appears on the semantic side as a 'Cantor theorem' for the category of Alexandroff spaces.

1 Introduction

Consider an extensional structure $\mathcal{U} \equiv \langle U; \in_{\mathcal{U}} \rangle$ for the language of set theory, and let us call a subset $A \subseteq U$ *collectable* if there exists $a \in U$ such that $A = \{x \in U \mid x \in_{\mathcal{U}} a\}$. We say that $\mathcal{U}$ is *topological* if there is a closure function $(\,\cdot\,)^\diamond : A \longmapsto A^\diamond$, defined on the set of *first-order definable* subsets of U, which satisfies the axioms of a topological closure, i. e.

$$\emptyset^\diamond = \emptyset, \qquad A \subseteq A^\diamond, \qquad A^{\diamond\diamond} = A^\diamond, \qquad (A \cup B)^\diamond = A^\diamond \cup B^\diamond,$$

and which is such that the collectable subsets of U are exactly the *closed* subsets, that is, those subsets $A \subseteq U$ satisfying $A^\diamond = A$.

The interest of topological structures lies in the fact that, even though not all definable subsets are collectable (Russell's paradox), each of these can at least be optimally *approximated* in such structures by the smallest set containing it. Furthermore, this appealing property can be expressed by a *first-order* axiom scheme that thus provides an approximated version of the naive comprehension scheme; it will be referred to in this paper as $(\diamond)$.

This alternative proposal for avoidance of the paradoxes was the core of Weydert's thesis [16], in which the author was led to prove the existence of very elaborated topological structures. These, subsequently called *hyperuniverses*, were independently discovered at the same time by Forti-Hinnion in [6] and have extensively been studied in different papers since then (see [7] for instance). The constructions given in [16] and [6] were actually both inspired by the pioneer work of Malitz [12] (see [6] for an historical account). Hyperuniverses were used to prove the consistency of the *positive set theory* $\mathrm{GPK}^+_{(\infty)}$, a natural extension of $(\diamond)$ which was deeply investigated in [4, 5].

Such an idea of approximation of naive comprehension originated however in a paper by Skala in [15], and then was refined by Manakos in [13].[1)] Skala proposed a pure axiomatic system in which any definable class has a least upper set approximation as well as a greatest lower one. She also added an axiom ensuring that any set has a complement, which we denote by $(\mathscr{C})$. Manakos dropped this last assumption in order to consider other possible extensions of his basic system which is designated by $(\triangle)$ in this paper. Those theories are characterized by the non-existence of '$\{a\}$' for some a.

* Corresponding author: e-mail: tlibert@ulb.ac.be
** e-mail: oesser@ulb.ac.be

1) It is interesting to note that Skala's paper is cited in the references of [16], and even in those of [12], but with no further comment.

Curiously, although semantic proofs of the consistency of (some extensions of) Skala's theory $(\triangle) + (\mathscr{C})$ appeared in [14, 10, 8], no *topological* attempt to characterize the models has been made. So we make the move in this paper by revisiting Skala's and Manakos' systems on both the axiomatic and semantic sides. The characterization we give (Theorem 6.2) shows that Skala's set theory is really weak in itself, but it also enables us to show that this theory is very adaptable: we prove that *any* universe of sets having singletons can be embedded in a model of $(\triangle)+(\mathscr{C})$ (Theorem 8.5). Another significant result of this paper is the incompatibility of $(\triangle)$ with the existence of '$\{x \mid a \in x\}$' for all a (Theorem 7.2), the semantic translation of which is a 'Cantor theorem' for the category of Alexandroff spaces (Section 9).

2 Notations

We denote by $\mathscr{L}$ the language in first-order predicate calculus *with equality* whose only non-logical primitive symbol is $\in$. Adding '$*$' as subscript means that $=$ is excluded from formulas.

When we introduce a formula φ as $\varphi(\bar{x})$, we just want to underline the free occurrences (possibly none) of each of the variables $\bar{x}$ in φ. Note that φ may have free variables other than $\bar{x}$. Those additional variables are called *parameters*. When we want to specify parameters too, we write $\varphi(\bar{x}, \bar{p})$ – in particular $\varphi(\bar{p})$ if $\bar{x}$ is empty. All the free variables of φ are then supposed to be among $\bar{x}, \bar{p}$.

Throughout the present paper, an $\mathscr{L}$-structure $\mathcal{U} \equiv \langle U; \in_{\mathcal{U}}\rangle$ will rather be looked at as $\langle U; [\,\cdot\,]_{\mathcal{U}}\rangle$, where $[\,\cdot\,]_{\mathcal{U}} : U \longrightarrow \mathscr{P}(U)$ is the corresponding *extension* function defined by $[v]_{\mathcal{U}} := \{u \in U \mid u \in_{\mathcal{U}} v\}$ for all $v \in U$. As usual, the interpretation of the equality in any structure is taken to be the identity, i. e. $u =_{\mathcal{U}} v$ if and only if u and v are denoting the same object in U. Note also that *extensionality* – Ext for short – is tacitly assumed throughout this paper, so that in all cases $[\,\cdot\,]_{\mathcal{U}}$ is injective.

Given a function $f : A \longrightarrow B$, $x \longmapsto f\text{'}x$, $\operatorname{rng} f$ will denote the range of f, i. e. $\operatorname{rng} f = \{f\text{'}x \mid x \in A\} \subseteq B$. Thus, given a set-theoretic structure $\mathcal{U}$, $\operatorname{rng}[\,\cdot\,]_{\mathcal{U}}$ is just the set of *collectable* subsets of U, and we let $\operatorname{def}[\,\cdot\,]_{\mathcal{U}}$ stand for the set of *definable* subsets of U, that is, those of the form $\{u \in U \mid \mathcal{U} \vDash \varphi(u, \bar{v})\}$ for some $\mathscr{L}$-formula $\varphi(x, \bar{p})$ and some assignment $\bar{v}$ in U of the parameters $\bar{p}$.

As usual, in developing our set theories, we will conveniently make use of *abstraction terms* '$\{x \mid \varphi(x)\}$' as a 'façon de parler'. So the rest of this section is devoted to introducing names for certain abstraction terms that are recurrent in our investigations. Let us also agree that whenever $\mathscr{X}$ is a name for '$\{x \mid \varphi(x)\}$', $(\mathscr{X})$ will stand for (the universal closure of) the φ-instance of the comprehension scheme, i. e. $\text{Comp}[\varphi(x)] :\equiv \exists y \forall x\,(x \in y \leftrightarrow \varphi)$.

Attached to propositional constants and atomic formulas are the following abstraction terms and their names:

$$\Lambda := \text{`}\{x \mid \bot\}\text{'},\;\; \mathrm{W} := \text{`}\{x \mid x \in x\}\text{'},\;\; \mathrm{V} := \text{`}\{x \mid \top\}\text{'},\;\; \mathscr{A}(p) := \text{`}\{x \mid p = x\}\text{'},\;\; \mathscr{B}(p) := \text{`}\{x \mid p \in x\}\text{'}\,.$$

We shall also introduce here the abstraction term corresponding to *complementation*:

$$\mathscr{C}(p) := \text{`}\{x \mid x \notin p\}\text{'}\,.$$

Beside $\in$ and $=$ two non-primitive binary relations deserve a particular interest. These are defined as follows:

$$x \,\dot{\leqslant}\, y \Leftrightarrow \forall z\,(z \in x \rightarrow z \in y), \qquad x \,\dot{\lessgtr}\, y \Leftrightarrow \forall z\,(x \in z \rightarrow y \in z).$$

The relationship between these and the primitive symbols $\in$ and $=$ is underlined by the following observation.

Fact 2.1 *$x \in y$ is equivalent to $\mathscr{A}(x) \,\dot{\leqslant}\, y$ and, assuming $(\mathscr{A})$, $x = y$ is equivalent to $x \,\dot{\lessgtr}\, y$.*

Whether $(\mathscr{A})$ is assumed or not, $\dot{\leqslant}$ has a major part in any set theory, as it is, of course, the *inclusion* relation, commonly denoted by $\subseteq$, but we will reserve this notation for the meta-theory. However odd that may seem, we are going to examine situations in which $(\mathscr{A})$ fails. According to Fact 2.1, in such a situation $\dot{\lessgtr}$ may be thought of as a *reminiscence* of the equality; this will be underscored in Sections 5 – 7.

We now introduce the following abstraction terms attached to $\dot{\leqslant}$ and $\dot{\lessgtr}$:

$$\mathscr{P}(p) := \text{`}\{x \mid x \,\dot{\leqslant}\, p\}\text{'}, \qquad \mathscr{M}(p) := \text{`}\{x \mid p \,\dot{\lessgtr}\, x\}\text{'}\,.$$

Of course, $\mathscr{P}(\,\cdot\,)$ is the well-known *power-set* operator, whereas the role of $\mathscr{M}(\,\cdot\,)$ is going to have to be explained. Neither '$\{x \mid p \,\dot{\leqslant}\, x\}$' nor '$\{x \mid x \,\dot{\lessgtr}\, p\}$' will be of interest to us in our investigations.

At last, we point out two useful abbreviations, which are sometimes referred to as the *extensional* and the *intensional* equalities respectively:

$$x \underset{\cdot}{=} y \Leftrightarrow x \underset{\cdot}{\leqslant} y \wedge y \underset{\cdot}{\leqslant} x, \qquad x \doteq y \Leftrightarrow x \mathrel{\dot{\leqslant}} y \wedge y \mathrel{\dot{\leqslant}} x.$$

In any extensional set theory, $=$ must coincide with $\underset{\cdot}{=}$, so that the equality might not be taken as primitive symbol. Note that, in the absence of $=$ in the language, extensionality should be formulated by $\forall x \forall y\, (x \underset{\cdot}{=} y \rightarrow x \doteq y)$, which guarantees that $\underset{\cdot}{=}$ has the substitutivity property for $\mathscr{L}_*$-formulas. It is worth noticing that the converse of this, namely $\forall x \forall y\, (x \doteq y \rightarrow x \underset{\cdot}{=} y)$, holds in any set theory satisfying $(\mathscr{B})$ and, still more obviously, in any one satisfying $(\mathscr{A})$, as then $\doteq$ must be $=$. In Section 7 we will reconsider this principle in a situation where both $(\mathscr{A})$ and $(\mathscr{B})$ fail.

3 The closure scheme

We are interested here in any *extensional* set theory based upon the following *approximation scheme*:

$(\Diamond)$ For every formula $\varphi(x)$, $\exists y\, (\forall x\, (\varphi \rightarrow x \in y) \wedge \forall z\, (\forall x\, (\varphi \rightarrow x \in z) \rightarrow y \underset{\cdot}{\leqslant} z))$.

In words, $(\Diamond)$ asserts the existence of a *least* set containing '$\{x \mid \varphi(x)\}$' for every formula $\varphi(x)$. We denote this (unique) set by $\{x \mid_\diamond \varphi\}$. If $\{x \mid_\diamond \varphi\} =$ '$\{x \mid \varphi(x)\}$', we shall say that φ is *continuous* w. r. t. x. Note that we will use that terminology in $(\Diamond)$-free context as well, so that φ is continuous w. r. t. x simply means that '$\{x \mid \varphi(x)\}$' is a set. A central question is then to know which formulas can be continuous, and in a type-free setting this may be split up into two parts, not independent of each other: which *atomic formulas* can be continuous, and which *logical connectives* can preserve the continuity of a formula – such connectives will be said to be *continuous* too.

Assuming $(\Diamond)$, $x = x$, $p = p$ and $\top$ are equally continuous w. r. t. x: they all define the universal set V. Notice also that $x \in p$ is clearly continuous w. r. t. x, seeing that $\{x \mid_\diamond x \in p\} = p$. The continuity of $p = x$ w. r. t. x is equivalent to the axiom $(\mathscr{A})$, and the one of $p \in x$ is $(\mathscr{B})$. These are going to be largely discussed. Finally, $x \in x$ may be continuous w. r. t. x only if $\neg$ is not continuous. And, of course, that $\neg$ preserves continuity is just $(\mathscr{C})$.

It is easy to see that $\wedge$ and $\forall$ are continuous under $(\Diamond)$ once we notice that this latter is equivalent to the following scheme of *definable intersections*:

$(\cap_\delta)$ For every formula $\psi(z)$, $\exists y \forall x\, (x \in y \leftrightarrow \forall z\, (\psi \rightarrow x \in z))$.

We denote this unique set y by $\bigcap$'$\{z \mid \psi(z)\}$'.

Proposition 3.1 *$(\Diamond)$ is equivalent to $(\cap_\delta)$.*

P r o o f. $\Rightarrow$: $\bigcap$'$\{z \mid \psi(z)\}$' $= \{x \mid_\diamond \forall z\, (\psi \rightarrow x \in z)\}$. $\Leftarrow$: $\{x \mid_\diamond \varphi\} = \bigcap$'$\{z \mid \forall x\, (\varphi \rightarrow x \in z)\}$'. □

Corollary 3.2 *Under $(\Diamond)$, $\wedge$ and $\forall$ preserve the continuity of a formula.*

P r o o f. Assume $(\cap_\delta)$ and suppose that $\varphi(x, y)$ is continuous w. r. t. x.
From '$\{x \mid \forall y\, \varphi(x, y)\}$' $= \bigcap$'$\{z \mid \exists y\, (z =$ '$\{x \mid \varphi(x, y)\}$'$)\}$', it follows that $\forall y\, \varphi(x, y)$ is continuous w. r. t. x, and that $\wedge$ preserves continuity results in the possibility of defining $p \cap q$ by $\bigcap$'$\{z \mid z = p \vee z = q\}$'. □

It is worth stressing that Proposition 3.1 shows that $(\Diamond)$ is just equivalent to particular instances of comprehension, namely $\mathrm{Comp}[\varphi(x)]$ where $\varphi(x)$ is of the form '$\forall z\, (\psi \rightarrow x \in z)$' for any $\mathscr{L}$-formula ψ (with the proviso that x does not occur free in ψ).

We now turn to an elementary characterization of the models of $(\Diamond)$. By a *closure operator on* $D \subseteq \mathscr{P}(U)$ we mean a $\subseteq$-preserving application $(\,\cdot\,)^\diamond : D \longrightarrow D$ such that $A \subseteq A^\diamond$ and $(A^\diamond)^\diamond = A$ for any $A \in D$. As usual, when $D = \mathscr{P}(U)$, we call $(\,\cdot\,)^\diamond$ a *closure operator on* U. A subset A of U is said to be *closed* if it lies in $\mathrm{rng}\,(\,\cdot\,)^\diamond$, that is, if $A^\diamond = A$. Given a structure $\mathcal{U}$, we also recall that $\mathrm{def}[\,\cdot\,]_\mathcal{U}$ stands for the set of definable subsets.

Theorem 3.3 *A set-theoretic structure $\mathcal{U}$ is a model of $(\Diamond)$ if and only if there exists a closure operator $(\,\cdot\,)^\diamond$ on $\mathrm{def}[\,\cdot\,]_\mathcal{U}$ such that $\mathrm{rng}\,[\,\cdot\,]_\mathcal{U} = \mathrm{rng}\,(\,\cdot\,)^\diamond$.*

P r o o f. We have $\{u \in U \mid \mathcal{U} \vDash \varphi(u,\bar{v})\}^{\diamond} = [\{x \mid_{\diamond} \varphi\}^{\mathcal{U}}(\bar{v})]_{\mathcal{U}}$ for any formula $\varphi(x,\bar{p})$ and $\bar{v}$ in U. □

This simple observation will enable us to easily answer natural questions on some extensions of $(\diamond)$ we are going to consider. Still more useful for constructing models is the second-order formulation of Theorem 3.3, in which $(\,\cdot\,)^{\diamond}$ is now a closure operator on U and $(\diamond)$ is replaced by its second-order version, namely

$$(\!(\diamond)\!) \qquad \forall P\,\exists y\,(\forall x\,(x \in P \to x \in y) \wedge \forall z\,(\forall x\,(x \in P \to x \in z) \to y \leqslant z)),$$

where P ranges over subsets of the domain of discourse. In the sequel, whenever we look at the second-order version of an axiom scheme, it will be designated with double parentheses '(( ))' and implicitly defined in the same way.

In all the examples we consider, it is even the case that $(\,\cdot\,)^{\diamond}$ is the closure operator associated with some (maybe trivial) *topology* on U, i. e. $(\,\cdot\,)^{\diamond}$ further satisfies $\emptyset^{\diamond} = \emptyset$ and $(A \cup B)^{\diamond} = A^{\diamond} \cup B^{\diamond}$ for any $A, B \subseteq U$. This is of course what has motivated the title of this paper. Notice that in such circumstances, Λ exists and $\vee$ is continuous too. Thus it remains to explore $(\mathscr{A})$, $(\mathscr{B})$, $(\mathscr{C})$ and the continuity of $\exists$ under $(\diamond)$.

Remark 3.4 The second-order version of Proposition 3.1 is the translation of the well-known correspondence between closure operators and (topped) intersection structures, which are complete lattices. Then, given a set-theoretic structure $\mathcal{U}$, what follows is immediate.

Fact 3.1 *If* $\mathcal{U} \vDash (\!(\cap_{\delta})\!)$, *then* $\langle U; \leqslant_{\mathcal{U}}\rangle$ *is a complete lattice.*

P r o o f. We observe that, for any $P \subseteq U$, $\bigwedge P := (\bigcap`\{z \mid z \in P\}')^{\mathcal{U}}$ is the infimum of P in $\langle U; \leqslant_{\mathcal{U}}\rangle$. □

We would just notify the reader that the converse of this does not hold. All that is possible to say if $\langle U; \leqslant_{\mathcal{U}}\rangle$ is a complete lattice is that, for any $P \subseteq U$, $\mathcal{U} \vDash \forall x\,(x \in \bigwedge P \to \forall z\,(z \in P \to z \in x))$.

4 Duality

The previous considerations naturally prompt us to examine the *symmetric* approach, according to which each class can be approximated from below. There is *a priori* no reason to prefer one view to the other. Besides, both views might equally well be taken simultaneously, which is going to be examined in the next sections.

The *dual* version of the approximation scheme $(\diamond)$ is defined as follows:

$(\Box)$ For every formula $\varphi(x)$, $\exists y\,(\forall x\,(x \in y \to \varphi) \wedge \forall z\,(\forall x\,(x \in z \to \varphi) \to z \leqslant y))$.

The unique y given by $(\Box)$ is designated by $\{x \mid_{\Box} \varphi\}$.

The scheme of definable intersections is now to be replaced by the one of *definable unions*:

$(\cup_{\delta})$ For every formula $\psi(z)$, $\exists y \forall x\,(x \in y \leftrightarrow \exists z\,(\psi \wedge x \in z))$.

We denote this unique y by $\bigcup`\{z \mid \psi(z)\}'$.

Proposition 4.1 $(\Box)$ *is equivalent to* $(\cup_{\delta})$.

P r o o f. $\Rightarrow$: $\bigcup`\{z \mid \psi(z)\}' = \{x \mid_{\Box} \exists z\,(\psi \wedge x \in z)\}$. $\Leftarrow$: $\{x \mid_{\Box} \varphi\} = \bigcup`\{z \mid \forall x\,(x \in z \to \varphi)\}'$. □

Corollary 4.2 *Under* $(\Box)$, $\vee$ *and* $\exists$ *preserve the continuity of a formula.*

In the same way, one could give a characterization of the models of $(\Box)$ in terms of interior operators.

Naturally, from the semantic point of view, this duality just consists in taking the complement: we define the dual $\mathcal{U}^{c}$ of a set-theoretic structure $\mathcal{U}$ to be $\langle U\,; (U \times U)\backslash \in_{\mathcal{U}}\rangle$, so that $\mathcal{U} \vDash (\diamond)\,/\,(\Box)$ if and only if $\mathcal{U}^{c} \vDash (\Box)\,/\,(\diamond)$. Assuming $(\mathscr{C})$, the duality is still more obvious on the axiomatic side:

Proposition 4.3 $(\diamond) + (\mathscr{C})$ *is equivalent to* $(\Box) + (\mathscr{C})$.

P r o o f. $\{x \mid_{\Box} \varphi\} = \mathscr{C}(\{x \mid_{\diamond} \neg\varphi\})$ and likewise by interchanging $\diamond$ and $\Box$. □

But if one is considering $(\mathscr{A})$, the symmetry is broken:

Fact 4.1 $(\Box)$ *implies* $non(\mathscr{A})$*, whereas* $(\Diamond) + (\mathscr{A})$ *is consistent.*

Proof. Using $(\cup_\delta)$, we define $r := \bigcup`\{z \mid \exists w\,(z = \mathscr{A}(w) \wedge w \notin w)\}'$. Clearly, we have $r \in r \rightarrow r \notin r$, and if $\mathscr{A}(r)$ existed, we would also have $r \notin r \rightarrow r \in r$. For the consistency of $(\Diamond) + (\mathscr{A})$, we rely on the existence of hyperuniverses (see [6] for instance). □

Despite the loss of singletons, we are going to be interested in any set theory based upon both $(\Diamond)$ and $(\Box)$. As mentioned in the introduction, this was initiated in [15], where $(\mathscr{C})$ is taken too, and then reinvestigated in [13] without assuming $(\mathscr{C})$ anymore.

5 The comprehension scheme revisited

By taking $(\Box)$, we must give up $(\mathscr{A})$. Nevertheless, we may use $(\Diamond)$ to define what we call *pseudo-singletons* or *molecules* (as named in [13]). By definition, the pseudo-singleton of p is the smallest set containing the singleton of p, that is, $\{x \mid_\diamond p = x\} = \bigcap \mathscr{B}(p)$, and this is actually nothing but $\mathscr{M}(p)$.

Proposition 5.1 $(\Diamond) + (\Box)$ *is equivalent to* $(\mathscr{M}) + (\Box)$.

Proof. ⇒: Obvious. ⇐: For any formula $\varphi(x)$, $\{x \mid_\diamond \varphi\} = \bigcup`\{z \mid \exists w\,(z = \mathscr{M}(w) \wedge \varphi(w))\}'$. □

The ⇐ part of the proof is interesting in that it shows that, under $(\Box)$,

(†) $\quad \{x \mid_\diamond \varphi\} = `\{x \mid \exists w\,(w \mathrel{\dot{\leqslant}} x \wedge \varphi(w))\}'$,

which leads to defining the scheme

$(\triangle)$ For every formula $\varphi(x)$, $\exists y \forall x\,(x \in y \leftrightarrow \exists w\,(w \mathrel{\dot{\leqslant}} x \wedge \varphi(w)))$.

Now can be shown

Proposition 5.2 $(\triangle)$ *is equivalent to* $(\Diamond) + (\Box)$.

Proof.
⇐: According to (†) above, just take y in $(\triangle)$ to be $\{x \mid_\diamond \varphi\}$.
⇒: If $\varphi(x)$ is taken to be $\exists z\,(x \in z \wedge \psi(z))$ in $(\triangle)$, what we get is just $(\cup_\delta)$. And if $\varphi(x)$ is taken to be $x = p$, we have $(\mathscr{M})$. □

The axiom scheme $(\triangle)$ should not leave the reader indifferent, for if $\dot{\leqslant}$ was the equality – which is the case if $(\mathscr{A})$ holds (cf. Fact 2.1) – it would be nothing but the *full* comprehension scheme.

Using Proposition 5.2, we can derive from Theorem 3.3 an elementary characterization of the models of $(\triangle)$. Given a preorder R on a set U, we denote by $(\,\cdot\,)^R$ the closure operator on U which is defined by

$$A^R := \{u \in U \mid \exists (a \in A)\,(a\,R\,u)\}\,.$$

Then, for any $D \subseteq \mathscr{P}(U)$, we let $D^{\uparrow}$ stand for $\{A^R \mid A \in D\}$ and $D^{\downarrow}$ for $\{A^{R^{-1}} \mid A \in D\}$, where R^{-1} is the opposite preorder. Now, given a set-theoretic structure $\mathcal{U}$, $\dot{\leqslant}_{\mathcal{U}}$ is a preorder on U, and as it is definable, we have $\mathrm{def}[\,\cdot\,]^{\uparrow}_{\mathcal{U}} \subseteq \mathrm{def}[\,\cdot\,]_{\mathcal{U}}$ and $\mathrm{def}[\,\cdot\,]^{\downarrow}_{\mathcal{U}} \subseteq \mathrm{def}[\,\cdot\,]_{\mathcal{U}}$, so that $(\,\cdot\,)^{\dot{\leqslant}_{\mathcal{U}}}$ is in particular a closure operator on $\mathrm{def}[\,\cdot\,]_{\mathcal{U}}$.

Theorem 5.3 *A model* $\mathcal{U}$ *of* $(\Diamond)$ *is a model of* $(\Box)$ *if and only if the closure operator* $(\,\cdot\,)^{\Diamond}$ *given by Theorem* 3.3 *coincides with* $(\,\cdot\,)^{\dot{\leqslant}_{\mathcal{U}}}$*, and thus* $\mathrm{rng}[\,\cdot\,]_{\mathcal{U}} = \mathrm{def}[\,\cdot\,]^{\uparrow}_{\mathcal{U}}$.

Proof. Assuming $(\Diamond)$, we saw above that $(\Box)$ is equivalent to $\{x \mid_\diamond \varphi\} = `\{x \mid \exists w\,(w \mathrel{\dot{\leqslant}} x \wedge \varphi(w))\}'$. □

Note that in any case, if $(\,\cdot\,)^{\diamond}$ was of the form $(\,\cdot\,)^R$ for some preorder R on U, this should be $\dot{\leqslant}_{\mathcal{U}}$: Suppose $u \mathrel{\dot{\leqslant}_{\mathcal{U}}} v$. As $\{u\}^R$ is closed, there exists $w \in U$ such that $[w]_{\mathcal{U}} = \{u\}^R$. Then, as $u \in \{u\}^R$, that is $u \in_{\mathcal{U}} w$, we get $v \in_{\mathcal{U}} w$, that is $v \in \{u\}^R$. Hence $u \; R \; v$. Conversely, suppose $u \; R \; v$ and let $w \in U$ such that $u \in_{\mathcal{U}} w$, that is $u \in [w]_{\mathcal{U}}$. Now, as $u \; R \; v$ and $[w]_{\mathcal{U}}$ is closed, i.e. $[w]_{\mathcal{U}}^R = [w]_{\mathcal{U}}$, we have $v \in [w]_{\mathcal{U}}$ too, that is $v \in_{\mathcal{U}} w$. Therefore $u \mathrel{\dot{\leqslant}_{\mathcal{U}}} v$.

Thus we may state the second-order version of Theorem 5.3 as follows.

Theorem 5.4 *A set-theoretic structure $\mathcal{U}$ is a model of $(\!(\triangle)\!)$ if and only if there exists a preorder R on U such that* $\operatorname{rng}[\,\cdot\,]_{\mathcal{U}} = \mathscr{P}(U)^{\uparrow}$. *In that case, R must coincide with $\dot{\leqslant}_{\mathcal{U}}$.*

With this, it is rather child's play to concoct models of $(\!(\triangle)\!)$, and in particular finite ones, which would testify to the weakness of $(\triangle)$ as a set theory on its own. Incidentally, there was even no need to invoke Theorem 5.4 to prove its consistency, as '$\{\Lambda, \mathrm{V}\}$' is obviously the simplest model, and this is also a model for the situation we examine hereafter.

6 The symmetric case

Without further assumptions, all the properties that $\dot{\leqslant}$ is certain to possess are *reflexivity* and *transitivity*. According to Fact 2.1, when $(\mathscr{A})$ fails, $\dot{\leqslant}$ may be thought of as a reminiscence of the equality. Then it would be legitimate to require the *symmetry* of $\dot{\leqslant}$. Under $(\triangle)$, this actually amounts to demanding that the negation preserves the continuity of a formula.

Proposition 6.1

(i) $(\mathscr{C})$ *implies* $(x \mathrel{\dot{\leqslant}} y \to y \mathrel{\dot{\leqslant}} x)$.

(ii) *Assuming* $(\triangle)$, $(x \mathrel{\dot{\leqslant}} y \to y \mathrel{\dot{\leqslant}} x)$ *implies* $(\mathscr{C})$.

Proof.

(i) Suppose $x \mathrel{\dot{\leqslant}} y$. If $x \notin z$, then $x \in \mathscr{C}(z)$, and so $y \in \mathscr{C}(z)$, that is $y \notin z$.

(ii) Suppose that $(x \mathrel{\dot{\leqslant}} y \to y \mathrel{\dot{\leqslant}} x)$. We show that $\{x \mid_{\diamond} x \notin p\} = \mathscr{C}(p)$: Let $x \in \{x \mid_{\diamond} x \notin p\}$. By $(\triangle)$, there exists w such that $w \mathrel{\dot{\leqslant}} x$ and $w \notin p$. Then $x \mathrel{\dot{\leqslant}} w$ and it follows that $x \notin p$ either, that is $x \in \mathscr{C}(p)$. □

To sum up, a model $\mathcal{U}$ of $(\triangle)$ satisfies $(\mathscr{C})$ if and only if $\dot{\leqslant}_{\mathcal{U}}$ is an equivalence relation, namely $\doteq_{\mathcal{U}}$. Thence we can give the next useful characterization which falls out of Theorem 5.4.

Theorem 6.2 *A set-theoretic structure $\mathcal{U}$ is a model of $(\!(\triangle)\!) + (\mathscr{C})$ if and only if there exists an equivalence relation R on U together with a bijection $f : U \longrightarrow \mathscr{P}(U/_R)$ such that $[u]_{\mathcal{U}} = \bigcup f\text{'}u$ for each $u \in U$. In that case, R must coincide with $\doteq_{\mathcal{U}}$.*

Proof. Observe that $(\,\cdot\,)^R$-closed subsets are just unions of equivalence classes, and thus we have $f\text{'}u = \{\{v\}^R \mid v \in [u]_{\mathcal{U}}\}$ for any $u \in U$. □

The simplified version that follows speaks for itself:

Theorem 6.3 *A set U is the universe of a model of $(\!(\triangle)\!) + (\mathscr{C})$ if and only if $|U| = 2^{\kappa}$ for some cardinal κ.*

Proof. The necessity follows directly from Theorem 6.2. For sufficiency, take any equivalence relation R on U such that $|U/_R| = \kappa$, and then any bijection $f : U \longrightarrow \mathscr{P}(U/_R)$ to define $[\,\cdot\,]_{\mathcal{U}}$ as in Theorem 6.2. □

At least, in assuming $(\triangle) + (\mathscr{C})$, all the basic logical connectives $\neg$, $\wedge$, $\vee$, $\forall$ and $\exists$ are continuous, but this results in serious drawbacks at the atomic level. We already know that $(\mathscr{A})$ is incompatible with $(\triangle)$. It turns out that $(\mathscr{B})$ is also incompatible with $(\triangle) + (\mathscr{C})$, and that this can even happen in a $(\triangle)$-free context:

Fact 6.1 *If $\neg$, $\wedge$, $\exists$ are continuous, then $(\mathscr{B})$ is inconsistent.*

Proof. Under the assumptions, $x \mathrel{\dot{\leqslant}} y \equiv \neg\exists z\,(x \in z \wedge y \notin z)$ is continuous w.r.t. x, and then, if we further assume that $y \in x$ is continuous w.r.t. x, so is $\exists y\,(x \mathrel{\dot{\leqslant}} y \wedge y \notin x)$. Now, let r be the set defined by this formula. It is easy to see that $r \in r$ if and only if $r \notin r$. □

Thus $(\mathscr{A})$, $(\mathscr{B})$, and obviously (W) are all incompatible with $(\triangle) + (\mathscr{C})$. We are going to show in the next section that, as for $(\mathscr{A})$, $(\mathscr{B})$ is already incompatible with $(\triangle)$ alone, but this is by far more subtle. To conclude and summarize, the next result – which is not mentioned in [15] or [13] – clearly states what the continuous formulas are under $(\triangle) + (\mathscr{C})$.

Theorem 6.4 *$(\triangle) + (\mathscr{C})$ is equivalent to Comp$[\varphi(x)]$ for any formula $\varphi(x)$ in $\mathscr{L}_*$, with the sole restriction that the abstracted variable x does not occur on the right-hand side of $\in$.*

P r o o f.
$\Rightarrow$: Follows from the continuity of all the logical connectives under $(\triangle) + (\mathscr{C})$ and the fact that the limitation to $\mathscr{L}_*$-formulas and the restriction on x prevent occurrences of $p = x$, $p \in x$, $x \in x$ at the atomic level.
$\Leftarrow$: Observe that $\bigcap$'$\{z \mid \psi(z)\}$' = '$\{x \mid \forall z\, (\psi \to x \in z)\}$' is just defined by such a suitable formula (note that as we assume extensionality, ψ can be reduced to an $\mathscr{L}_*$-formula by replacing any occurrence of $=$ by $\underset{\cdot}{=}$). Thus we get $(\cap_\delta)$. And that we have $(\mathscr{C})$ is still more obvious. □

It should be stressed that, as well as the absence of $=$, this simple restriction on the abstracted variable has disastrous effects on the usual development of set theory, as for instance it prevents us from defining the power-set of any given set p, seeing that $\mathscr{P}(p) =$ '$\{x \mid \forall z\, (z \in x \to z \in p)\}$'.

7 The antisymmetric case

The absence of $(\mathscr{C})$ seems to leave the door open to the continuity of $x \in x$ and $p \in x$ w. r. t. x. In fact, that (W) is compatible with $(\triangle)$ is easily seen:

Example 7.1 Take $U := \{a, b, c\}$ with the relation $R := (U \times U) \setminus \{(b, a), (c, a)\}$, i. e. $\{a\}^R = \{a, b, c\}$, $\{b\}^R = \{b, c\} = \{c\}^R$ (note that R is neither symmetric nor antisymmetric), and define $[\,\cdot\,]_{\mathcal{U}}$ as follows: $[a]_{\mathcal{U}} := \emptyset$, $[b]_{\mathcal{U}} := \{b, c\}$, $[c]_{\mathcal{U}} := U$. Thus, $\mathcal{U}$ is a model of $(\!(\triangle)\!)$ in which a is Λ, b is W, and c is V. Notice that both $\mathscr{B}(b)$ and $\mathscr{B}(c)$ exist and are equal to b, but $\mathscr{B}(a)$ does not exist in $\mathcal{U}$.

Trying to find a model of $(\mathscr{B})$ directly is a less obvious task for it is easily seen that there is no finite model of $(\mathscr{B})$ (see [11]). Under $(\triangle)$, the task will soon appear vain. To proceed, we first point out the following result (to be compared with Proposition 6.1).

Proposition 7.1

(i) *$(\mathscr{B})$ implies $(x \dot{\leqslant} y \to x \underset{\cdot}{\leqslant} y)$.*

(ii) *Assuming $(\triangle)$, $(x \dot{\leqslant} y \to x \underset{\cdot}{\leqslant} y)$ implies $(\mathscr{B})$.*

P r o o f.
(i) Suppose $x \dot{\leqslant} y$. If $z \in x$, then $x \in \mathscr{B}(z)$, and so $y \in \mathscr{B}(z)$, that is $z \in y$.
(ii) Suppose that $(x \dot{\leqslant} y \to y \underset{\cdot}{\leqslant} x)$. We show that $\{x \mid_\diamond p \in x\} = \mathscr{B}(p)$: Let $x \in \{x \mid_\diamond p \in x\}$. By $(\triangle)$, there exists w such that $w \dot{\leqslant} x$ and $p \in w$. Then $w \underset{\cdot}{\leqslant} x$ and it follows that $p \in x$ too, that is $x \in \mathscr{B}(p)$. □

We now prove one of the main results of this paper.

Theorem 7.2 *$(x \dot{\leqslant} y \to x \underset{\cdot}{\leqslant} y)$ – and so $(\mathscr{B})$ – is incompatible with $(\triangle)$.*

P r o o f.[2] Assume $(\triangle) + (x \dot{\leqslant} y \to x \underset{\cdot}{\leqslant} y)$, and let $x \in' y$ stand in what follows for $\forall z\, (x \in z \to z \in y)$. It is easy to see that the existence of $\mathscr{C}'(p) :=$ '$\{x \mid x \underset{\cdot}{\nleqslant} p\}$' for any p follows from $(\triangle) + (x \dot{\leqslant} y \to x \underset{\cdot}{\leqslant} y)$, and we observe that

$$(*) \qquad \forall x\, (x \in' \mathscr{C}'(p) \leftrightarrow x \notin p).$$

Indeed, if $x \in' \mathscr{C}'(p)$, then $\forall z\, (x \in z \to z \underset{\cdot}{\nleqslant} p)$, and as $p \underset{\cdot}{\leqslant} p$, we have $x \notin p$. Conversely, if $x \notin p$, then clearly $\forall z\, (x \in z \to z \underset{\cdot}{\nleqslant} p)$, and this is $x \in' \mathscr{C}'(p)$.
Now, take $p :=$ '$\{x \mid \exists w\, (w \dot{\leqslant} x \wedge w \in' w)\}$', whose existence follows from $(\triangle)$, and then let $r := \mathscr{C}'(p)$. From $(*)$ above results that $\forall x\, (x \in' r \leftrightarrow \forall w\, (w \dot{\leqslant} x \to w \notin' w))$. In particular, we have $\forall x\, (x \in' r \to x \notin' x)$.

[2] A semantic proof of this result appeared in [11]. Its syntactical translation presented here is due to Marcel Crabbé.

So we must have $r \notin' r$. But then there exists w_0 such that $w_0 \dot{\leqslant} r$ and $w_0 \in' w_0$. As we have assumed $(x \dot{\leqslant} y \to x \underset{\cdot}{\leqslant} y)$, we thus have $w_0 \underset{\cdot}{\leqslant} r$ and $w_0 \in' w_0$, from which it follows that $w_0 \in' r$, and then that $w_0 \notin' w_0$ as well. A contradiction. □

A more direct proof can be given but for a slight variant of Theorem 7.2:

Proposition 7.3 $(x \dot{\leqslant} y \to y \underset{\cdot}{\leqslant} x)$ *is incompatible with* $(\triangle)$.

P r o o f. Let $r := `\{x \mid \exists w\,(w \dot{\leqslant} x \wedge w \notin w)\}'$, whose existence follows from $(\triangle)$. Clearly $r \notin r \to r \in r$, so we must have $r \in r$. Then there exists w_0 such that $w_0 \dot{\leqslant} r$ and $w_0 \notin w_0$. If we assume $(x \dot{\leqslant} y \to y \underset{\cdot}{\leqslant} x)$, we get $w_0 \notin r$, which is impossible for $w_0 \in r$ (since $w_0 \notin w_0$). □

We also mention that the converse of $(x \dot{\leqslant} y \to x \underset{\cdot}{\leqslant} y)$ is obviously consistent with $(\triangle)$, as it holds in the two-points model $`\{\Lambda, \mathrm{V}\}'$ or even in the model given in Example 7.1. However, this is a rather odd principle, as, for instance, it is apparent that $`\{\Lambda\}'$ cannot exist under $(x \underset{\cdot}{\leqslant} y \to x \dot{\leqslant} y)$. A natural example of a set theory satisfying this is investigated in [9, 11].

In assuming $(\mathscr{B})$ (and Ext), $\dot{\leqslant}$ is *anti-symmetric*, i. e. $(x \doteq y \to x = y)$. It is then legitimate to enquire whether this familiar principle could not be compatible with $(\triangle)$. At least, it would prevent the existence of finite models:

Fact 7.1 *Any model* $\mathcal{U}$ *of* $(\Lambda) + (\mathscr{M}) + (x \doteq y \to x = y)$ *is infinite.*

P r o o f. Notice that $x \doteq y$ is equivalent to $\mathscr{M}(x) = \mathscr{M}(y)$ and that $\Lambda = \mathscr{M}(x)$ for no x. Then it is clear that, under the assumptions, we can define a potentially infinite sequence of elements in U by iterating $\mathscr{M}(\,\cdot\,)$ from $\Lambda^{\mathcal{U}}$. □

The question of consistency of $(\triangle) + (x \doteq y \to x = y)$ is raised and left open in [13]. By invoking Theorem 5.4, it is now fairly easy to give a positive answer.

Example 7.2 Take $U := \mathbb{N}$ (the set of natural numbers), R the usual ordering $\leqslant$ on $\mathbb{N}$, and define $[\,\cdot\,]_{\mathcal{U}}$ by

$$[0]_{\mathcal{U}} := \emptyset, \qquad [n]_{\mathcal{U}} := \{m \mid n-1 \leqslant m\} \quad \text{for any } n \geqslant 1.$$

Thus it is clear that $\operatorname{rng}[\,\cdot\,]_{\mathcal{U}}$ is just the collection of $(\,\cdot\,)^R$-closed subsets of U, so $\mathcal{U} \vDash ((\triangle))$. In $\mathcal{U}$, 0 is Λ, 1 is V, and 2 is W, which shows that this latter can exist when $\dot{\leqslant}$ is antisymmetric. On the other hand, note that $\mathscr{B}(u)$ exists for no u in $\mathcal{U}$.

8 Normality

Assuming $(\Box)$, not every singleton can exist, so we shall say that a set x is *normal* if $`\{x\}'$ exists. No surprises, as we show in this section, very few things can be said about the class of normal sets, which is defined by

$$\mathscr{N} := `\{x \mid \exists y\,(y = \mathscr{A}(x))\}' .$$

Proposition 8.1 $\exists x\,(x \notin \mathscr{N})$.

P r o o f. This is just a reformulation of the first part of Fact 4.1. □

On the other hand, $\exists x\,(x \in \mathscr{N})$ is not even derivable from $(\triangle) + (\mathscr{C})$, for in the two-points model $`\{\Lambda, \mathrm{V}\}'$ we have $\mathscr{N} = \Lambda$. As certified by this example, it can be shown, at least, that $\mathscr{N}$ is always a set.

Lemma 8.2 *For any formula* $\varphi(x)$, *if* $z \in \mathscr{N}$ *and* $\varphi(z)$, *then* $z \in \{x \mid_{\Box} \varphi\}$.

P r o o f. Suppose $\varphi(z)$ and $z \in \mathscr{N}$. Then, $\{x \mid_{\Box} \varphi\} \cup \mathscr{A}(z)$ is a set, and as $\forall x\,(x \in \{x \mid_{\Box} \varphi\} \cup \mathscr{A}(z) \to \varphi)$, it follows that $\{x \mid_{\Box} \varphi\} \cup \mathscr{A}(z) \underset{\cdot}{\leqslant} \{x \mid_{\Box} \varphi\}$. Hence $z \in \{x \mid_{\Box} \varphi\}$. □

Proposition 8.3 $\mathscr{N}$ *is a set.*

P r o o f. Take $\varphi(x)$ to be $\exists y\,(y = \mathscr{A}(x))$ in the previous lemma. □

Such a simple question as to know whether $\mathscr{N} \in \mathscr{N}$ or not is undecidable. In the two-points model, we have $\mathscr{N} = \Lambda$, so that $\mathscr{N} \notin \mathscr{N}$ is consistent; and by invoking Theorem 6.2, it is easy to concoct a model $\mathcal{U}$ of $(\triangle) + (\mathscr{C})$ in which, for instance, $\mathscr{N} = \mathscr{A}(\mathscr{N})$, so that $\mathscr{N} \in \mathscr{N}$ is also consistent.

Example 8.1 Take $U = \{a, b, c, d\}$, with $\{d\}$ and $\{a, b, c\}$ as R-classes, and with $[a]_{\mathcal{U}} := \emptyset$, $[b]_{\mathcal{U}} := \{a, b, c\}$, $[c]_{\mathcal{U}} := U$, $[d]_{\mathcal{U}} := \{d\}$. In $\mathcal{U}$, d is $\mathscr{N}$.

With the help of Lemma 8.2, we can also give an eloquent characterization of abnormal sets. These are just *tokens* of the discontinuity of a formula, in the following sense:

Proposition 8.4 *$z \notin \mathscr{N}$ if and only if there exists a formula $\varphi(x)$ such that $\varphi(z)$ but $z \notin \{x \mid_{\square} \varphi\}$.*

P r o o f. For necessity, let $z \notin \mathscr{N}$. Then just take $\varphi(x)$ to be $z = x$. For sufficiency, use Lemma 8.2. □

Perhaps the most significant manifestation of the absence of control over $\mathscr{N}$ is the following positive result, which states that $\mathscr{N}$ may be taken to be *any* preexisting universe of normal sets (e. g. a model of ZF as in [8]).

Theorem 8.5 *Let $\mathcal{V}$ be any infinite extensional set-theoretic structure satisfying $(\mathscr{A})$. Then there exists a model $\mathcal{U}$ of $(\!(\triangle)\!) + (\mathscr{C})$ such that:*

(1) $[\,\cdot\,]_{\mathcal{V}} = [\,\cdot\,]_{\mathcal{U}}$ *restricted on V;*

(2) $V = \{u \in U \mid \mathcal{U} \vDash u \in \mathscr{N}\}$*;*

(3) $\mathscr{P}(V) \subseteq \operatorname{rng}[\,\cdot\,]_{\mathcal{U}}$.

P r o o f. Let U be $V \cup V' \cup V''$, where $V' := \mathscr{P}(V) \setminus \operatorname{rng}[\,\cdot\,]_{\mathcal{V}}$ and V'' is any set of cardinality $2^{|V|}$, and where we also assume that V, V', V'' are pairwise disjoint. We now equip each of these with an equivalence relation: we define S on V by $V/_S := \{\{v\} \mid v \in V\}$, S' on V' by $V'/_{S'} := \{V'\}$, and we take any equivalence S'' on V'' such that $|V''/_{S''}| = |V|$ and $|K| \geqslant 2$ for all $K \in V''/_{S''}$. Then we let R stand for the equivalence on U defined by $S \cup S' \cup S''$. Notice that $|U| = |\mathscr{P}(U/_R)| = 2^{|V|}$. Thus, if we set $f\text{'}v := \{\{w\} \mid w \in_{\mathcal{V}} v\}$ for each $v \in V$ and $f\text{'}W := \{\{w\} \mid w \in W\}$ for all $W \in V'$, then f can be extended over U as to define a bijection $U \longrightarrow \mathscr{P}(U/_R)$. By invoking Theorem 6.2, we can now turn U into a set-theoretic structure $\mathcal{U} \vDash (\!(\triangle)\!) + (\mathscr{C})$ by setting $[u]_{\mathcal{U}} := \bigcup f\text{'}u$ for any $u \in U$. It remains to check that $\mathcal{U}$ satisfies (1), (2), (3). For any $v \in V$, we have $[v]_{\mathcal{U}} = \bigcup\{\{w\} \mid w \in_{\mathcal{V}} v\} = \{w \mid w \in_{\mathcal{V}} v\}$, and this is just $[v]_{\mathcal{V}}$. In particular, since $\mathcal{V} \vDash (\mathscr{A})$, we have $\{v\} = [\mathscr{A}^{\mathcal{V}}(v)]_{\mathcal{V}} = [\mathscr{A}^{\mathcal{V}}(v)]_{\mathcal{U}}$, which shows that $\mathcal{U} \vDash v \in \mathscr{N}$ and that $\mathscr{A}^{\mathcal{U}}(v) = \mathscr{A}^{\mathcal{V}}(v)$ for any $v \in V$. Conversely, suppose $\mathcal{U} \vDash u \in \mathscr{N}$, that is $\{u\} = \bigcup A$ for some $A \subseteq U/_R$. If we had $A \cap (V'/_{S'} \cup V''/_{S''}) \neq \emptyset$, we would have $|\bigcup A| \geqslant 2$, for V' is infinite and $|K| \geqslant 2$ for any $K \in V''/_{S''}$. Therefore we must have $A \subseteq V/_S$, and so $\{u\} \subseteq V$. Finally, let $W \subseteq \mathscr{P}(V)$. If $W \in \operatorname{rng}[\,\cdot\,]_{\mathcal{V}}$, then clearly $W \in \operatorname{rng}[\,\cdot\,]_{\mathcal{U}}$. If $W \notin \operatorname{rng}[\,\cdot\,]_{\mathcal{V}}$, $W \in V'$ and then $[W]_{\mathcal{U}} = \bigcup f\text{'}W = W$, so $W \in \operatorname{rng}[\,\cdot\,]_{\mathcal{U}}$. □

In particular, Theorem 8.5 shows that *any* extension of $(\triangle) + (\mathscr{C})$ by a consistent set of axioms characterizing $\mathscr{N}$ is itself consistent. So Skala's theory is intrinsically weak but very adaptable.

One might not be happy with the arbitrary nature of the model constructed in the proof of Theorem 8.5. A possible way to try to define a notion of coherence in models of $(\triangle) + (\mathscr{C})$ is discussed in the next and last section, in which we summarize the mathematical content of this paper.

9 Coherence

Given a topological space U, let $\mathscr{P}_{\mathrm{cl}}(U)$ stand for the set of *closed* subsets of U, and $\mathscr{P}_{\mathrm{op}}(U)$ for the set of *open* ones. In what follows we use $\simeq$ to indicate that there is a bijection between two sets, and $\cong$ when we require the bijection to be an homeomorphism.

According to the second-order version of Theorem 3.3, resp. its dual, any topological space U such that $U \simeq \mathscr{P}_{\mathrm{cl}}(U)$, resp. $U \simeq \mathscr{P}_{\mathrm{op}}(U)$, gives rise to a model of $(\!(\Diamond)\!)$, resp. $(\!(\Box)\!)$. Notice that the closure/interior operator attached to a given model of $(\!(\Diamond)\!)/(\!(\Box)\!)$ need not be topological. For our purposes, however, we shall restrict ourselves to topological models. But even in that case there are still many insignificant models of $(\!(\Diamond)\!)/(\!(\Box)\!)$, in particular finite ones. So in what remains of this paper we shall rather be concerned with the existence of solutions to $U \cong \mathscr{P}_{\mathrm{cl}}(U)/U \cong \mathscr{P}_{\mathrm{op}}(U)$ within specific categories of topological spaces. As we shall see, the existence of such solutions is closely related to the consistency problem of some natural extensions of $(\!(\Diamond)\!)/(\!(\Box)\!)$. It may also be said that such solutions $\mathcal{U}$ to $U \cong \mathscr{P}_{\mathrm{cl}}(U)/U \cong \mathscr{P}_{\mathrm{op}}(U)$ define *natural* models for $(\!(\Diamond)\!)/(\!(\Box)\!)$, in that the extension function $[\,\cdot\,]_{\mathcal{U}}$ is to be not only a bijection but a homeomorphism. This does guarantee that there is some coherence in the process of assigning extensions to sets – it is understood here that $\mathscr{P}_{\mathrm{cl}}(U)/\mathscr{P}_{\mathrm{op}}(U)$ has itself been equipped with a *natural* topological structure derived from the one of U. Examples of such solutions are provided by hyperuniverses, which are compact T_2 (κ-)topological spaces solutions to $U \cong \mathscr{P}_{\mathrm{cl}}(U)$, where $\mathscr{P}_{\mathrm{cl}}(U)$ is endowed with the (κ-)Vietoris topology (and where κ is a weakly compact cardinal – see [6, 7]). As mentioned in the introduction, these turned out to be models of the positive theory $\mathrm{GPK}^+_{(\infty)}$, a natural extension of $(\Diamond)$ (see [4]).

9.1 Alexandroff spaces

Any model of $(\!(\triangle)\!)$ is topological. This is a direct consequence of Theorem 5.4 which states that such a model appears as a preordered set $\langle U; R\rangle$ such that $U \simeq \mathscr{P}(U)^{\uparrow}$, and this latter is just $\mathscr{P}_{\mathrm{op}}(U)$ if U is endowed with the *Alexandroff* topology. Notice that, as $\mathscr{P}(U)^{\uparrow} = \mathscr{P}(U^*)^{\downarrow}$ where U^* is $\langle U; R^{-1}\rangle$, any model of $(\!(\triangle)\!)$ may equally be viewed as a topological solution to $U \simeq \mathscr{P}_{\mathrm{cl}}(U)$, where U is now endowed with the Alexandroff topology of U^*. This is already implicitly contained in Theorem 5.4, for $(\,\cdot\,)^R$ is a closure operator, not an interior one. Still, it is more natural to view $\mathscr{P}(U)^{\uparrow}$ as $\mathscr{P}_{\mathrm{op}}(U)$. The reason is that then R – and so $\dot{\leqslant}_{\mathcal{U}}$ – coincides with the *specialization preorder* $\lhd_U$ of the topology, which is defined by

$$u \lhd_U v \Leftrightarrow \forall (A \in \mathscr{P}_{\mathrm{op}}(U))\,(u \in A \to v \in A).$$

This is also referred to as the *indiscernibility relation associated with the topology*, seeing that $u \not\lhd_U v$ if and only if $\exists (A \in \mathscr{P}_{\mathrm{op}}(U))\,(u \in A \wedge v \notin A)$, which means that '$u$ is *discernible* from v', in topological terms.

It is clear that for any topological space U we have $\mathscr{P}_{\mathrm{op}}(U) \subseteq \mathscr{P}(U)^{\uparrow}$, where this latter is taken with respect to $\lhd_U$. And then it is easy to see that a topological space U will satisfy $\mathscr{P}_{\mathrm{op}}(U) = \mathscr{P}(U)^{\uparrow}$ if and only if $\mathscr{P}_{\mathrm{op}}(U)$ is closed under arbitrary intersections, that is to say, if this latter also defines the set of closed subsets for some topology on U. Those topological spaces are called *Alexandroff spaces*. Their topology is thus generated by a preorder, which must coincide with the specialization preorder.

All that is to say that a model of $(\!(\triangle)\!)$ is exactly a solution $\mathcal{U}$ to $U \simeq \mathscr{P}_{\mathrm{op}}(U)$ within ALEX, the category of Alexandroff spaces. Notice that the Alexandroff-continuous functions are just the functions preserving the indiscernibility relation.

Given an Alexandroff space U, a possible way to transfer the indiscernibility relation onto $\mathscr{P}(U)$, and thus to turn this latter into an Alexandroff space as well, is by defining $A \lhd_{\mathscr{P}(U)} B$ if and only if $A \subseteq B^{\lhd_U}$. The restriction of this to $\mathscr{P}_{\mathrm{op}}(U) = \mathscr{P}(U)^{\uparrow}$ is nothing but the inclusion relation $\subseteq$. In words, A is discernible from B in $\mathscr{P}_{\mathrm{op}}(U)$ if and only if there exists $a \in A$ with $a \notin B$. We now remark that, with respect to that Alexandroff topology on $\mathscr{P}_{\mathrm{op}}(U)$, the equation $U \cong \mathscr{P}_{\mathrm{op}}(U)$ is *unsolvable* within ALEX. This directly follows from the incompatibility of $(\mathscr{B})$ with $(\triangle)$, Theorem 7.2, whose semantical translation thus appears as a 'Cantor theorem' for ALEX.

Another possible way to see $\mathscr{P}(U)$ as an Alexandroff space is by defining $A \lhd_{\mathscr{P}(U)} B$ if and only if $B \subseteq A^{\lhd_U}$, which may even seem more natural, for then $U \longrightarrow \mathscr{P}(U)$, $u \longmapsto \{u\}$ is Alexandroff-continuous. The restriction of this to $\mathscr{P}_{\mathrm{op}}(U)$ is now the reverse inclusion $\supseteq$, and that $U \cong \mathscr{P}_{\mathrm{op}}(U)$ is still unsolvable within ALEX follows from Proposition 7.3.

Likewise, one can show that the equation $U \cong \mathscr{P}_{\mathrm{cl}}(U)$, whether this latter is equipped with $\subseteq$ or $\supseteq$, is unsolvable within ALEX. Incidentally, a proof of this in the case where $\mathscr{P}_{\mathrm{cl}}(U)$ is equipped with $\subseteq$ was originally given in [3], where it was shown that, given an *ordered* set U, there is no surjective monotone function $U \longrightarrow \mathscr{P}(U)^{\downarrow}$.

Remark 9.1 Note that both $U \cong \mathscr{P}_{\mathrm{op}}(U)$ and $U \cong \mathscr{P}_{\mathrm{cl}}(U)$ are solvable within SCOTT, the category of *Scott spaces*, which are just the dcpo's endowed with the Scott-topology (for which the specialization preorder then coincides with the dcpo ordering). Interestingly, it is proved in [11] that such solutions yield natural models for *positive abstraction*, another variant of positive set theory.

9.2 Quasi-discrete spaces

If the specialization preorder of an Alexandroff space U is *symmetric*, i. e. it is an equivalence relation, we say that U is a *quasi-discrete space*. It is very easy to see that a topological space U is quasi-discrete if and only if $\mathscr{P}_{\text{op}}(U) = \mathscr{P}_{\text{cl}}(U)$. In words, the topology of a quasi-discrete space is generated by an equivalence relation: the open subsets, as well as the closed ones, are just unions of equivalence classes. We call the category of quasi-discrete spaces QUASI (this is a sub-category of ALEX). According to Theorem 6.2, a model of $(\!(\triangle)\!) + (\mathscr{C})$ is just a solution $\mathcal{U}$ to $U \simeq \mathscr{P}_{op}(U)$ – equally $U \simeq \mathscr{P}_{cl}(U)$ – within QUASI. In such an $\mathcal{U}$, $\doteq_{\mathcal{U}}$ coincides with $\lhd_U$.

Now, given a quasi-discrete space U, the appropriate way to enrol $\mathscr{P}(U)$ in QUASI is by defining $A \lhd_{\mathscr{P}(U)} B$ if and only if $A \subseteq B^{\lhd_U}$ *and* $B \subseteq A^{\lhd_U}$, i. e. if and only if $A^{\lhd_U} = B^{\lhd_U}$, which is thus an equivalence relation on $\mathscr{P}(U)$. But the restriction of this to $\mathscr{P}_{\text{op}}(U)$ $(= \mathscr{P}_{\text{cl}}(U))$ is the identity, so that, for obvious reasons here, there is no solution $\mathcal{U}$ to $U \cong \mathscr{P}_{\text{op}}(U)$ within QUASI (such an $\mathcal{U}$ would be a model of $(\triangle) + (\mathscr{C}) + (x \doteq y \rightarrow x = y)$).

Nevertheless, it is possible to define a *consistent* notion of coherence in models of $(\!(\triangle)\!) + (\mathscr{C})$. We shall say that a solution $\mathcal{U}$ to $U \simeq \mathscr{P}_{\text{op}}(U)$ within QUASI is *acceptable* if for any indiscernible u, v in U – that is $u \lhd_U v$ – either $[u]_{\mathcal{U}} = [v]_{\mathcal{U}} = \emptyset$ or $[u]_{\mathcal{U}} \cap [v]_{\mathcal{U}} \neq \emptyset$. This clearly vouches for a certain coherence in the process of assigning extensions to sets. The first-order translation of this condition on the axiomatic side is expressed by $(x \doteq y \rightarrow x \between y)$, where $x \between y :\equiv (x = \Lambda \wedge y = \Lambda) \vee \exists z\, (z \in x \wedge z \in y)$.

At least, any acceptable solution is infinite, as the next observation shows:

Fact 9.1 *Any model* $\mathcal{U}$ *of* $(\Lambda) + (\mathscr{M}) + (\mathscr{C}) + (x \doteq y \rightarrow x \between y)$ *is infinite.*

P r o o f. First notice that $\Lambda \between \mathscr{M}(x)$ for no x, and that, as $\dot{\leqslant}$ coincides with $\doteq$ under $(\mathscr{C})$, we have that $\mathscr{M}(x) \between \mathscr{M}(y)$ implies $x \doteq y$. Then it follows from the assumptions that we can define a potentially infinite sequence of elements in U by iterating $\mathscr{M}(\,\cdot\,)$ from $\Lambda^{\mathcal{U}}$ (as in Fact 7.1). □

Just to stress the combinatoric nature of seeking an acceptable solution, we now formulate the corresponding version of Theorem 6.2.

Fact 9.2 *A set* U *is the universe of an acceptable solution if and only if there is an equivalence relation* R *on* U *together with a bijection* $f : U \longrightarrow \mathscr{P}(U/_R)$ *such that from* $u\ R\ v$ *follows* $f\text{'}u \cap f\text{'}v \neq \emptyset$ *or* $f\text{'}u = f\text{'}v = \emptyset$.

Interestingly, the existence of an acceptable solution was established in [1], but without any reference to Skala's set theory. The authors used that structure to promote what is called 'rough set theory' (see also [2]).

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