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Irrational Numbers

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(a) In Section 5 we saw that the third car always develops wild acceleration if $C > \pi/2$.

(b) By Q9(c) we see that no car *ever* develops wild acceleration, but each car is wilder than its predecessor if $C > \omega/2\sin\omega$.

References

1. F. Haight, Mathematical Theories of Traffic Flow, Academic Press, New York, 1963.
2. D. Gazis, Mathematical theory of automobile traffic, Science, 157 (7–21–67) 273–281.
3. R. Herman and Gardels, Vehicular traffic flow, Scientific American, 209 no. 6 (Dec. 1963) 35–43.
4. R. Herman, Montroll, Potts, and Rothery, Traffic dynamics: Analysis of stability in car following, Operations Res., 7 (1959) 86–103.
5. R. Chandler, Herman, and Montroll, Traffic Dynamics: Studies in car following, Operations Res., 6 (1958) 165–184.
6. H. Simon, Models of Man, Chapter 13, Application of Servomechanism Theory to Production Control, Wiley, New York, 1957.

IRRATIONAL NUMBERS

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For the past few years a clever proof has been making the rounds of the various mathematics departments.

Theorem 1. *An irrational number raised to an irrational power may be rational.*

Proof: Consider the identity

$$[\sqrt{2}^{\sqrt{2}}]^{\sqrt{2}} = 2.$$

If $\sqrt{2}^{\sqrt{2}}$ is rational then we are finished. If not then $\sqrt{2}^{\sqrt{2}}$ is irrational so $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ is the example.

This proof seems first to have been published by Dov Jarden as a curiosity in [3]. The proof was published again in [2]. Note that while the proof is elementary, it is non-constructive. The non-constructivity enters in the form of the logical principle of the excluded middle (*tertium non datur*) which the intuitionists reject.

Actually $\sqrt{2}^{\sqrt{2}}$ is irrational, being the square root of Hilbert's number $2^{\sqrt{2}}$, proved transcendental by Kuzmin [1] in 1930. But this result, which is not elementary, is not used above. Only the irrationality of $\sqrt{2}$ is used.

Consider next the related theorem.

Theorem 2. *An irrational number raised to an irrational power may be irrational.*

Of course we can use set theoretical principles to prove that a^b is irrational for almost all real numbers b. Or we can use the result of Kuzmin [1] to prove Theorem 2 But does Theorem 2 have an elementary proof?

Proof: Consider the identity $\sqrt{2}^{(\sqrt{2}+1)} = (\sqrt{2}^{\sqrt{2}})\sqrt{2}$.

If $\sqrt{2}^{\sqrt{2}}$ is irrational then we are finished. If not, then $\sqrt{2}^{\sqrt{2}}$ is rational. Hence $(\sqrt{2}^{\sqrt{2}})\sqrt{2}$ is irrational, and $\sqrt{2}^{(\sqrt{2}+1)}$ is the example in this case.

There is also a simple identity by means of which it can be proved that a rational number raised to an irrational power may be irrational. But perhaps the reader would enjoy finding this one himself.

References

1. R. Kuzmin, On a new class of transcendental numbers, Izv. Akad. Nauk SSSR, Ser. Mat., 7 (1930) 585–597.
2. Mathematics Magazine, 39(1966) 111, 134.
3. Scripta Mathematica, 19 (1953) 229.

A SIMPLE PROOF OF THE FORMULA $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$

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Start with the inequality $\sin x < x < \tan x$ for $0 < x < \pi/2$, take reciprocals, and square each member to obtain

$$\cot^2 x < 1/x^2 < 1 + \cot^2 x.$$

Now put $x = k\pi/(2m+1)$ where k and m are integers, $1 \leqq k \leqq m$, and sum on k to obtain

$$\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} < m + \sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1}. \tag{1}$$

But since we have

$$\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} = \frac{m(2m-1)}{3}, \tag{2}$$

(a proof of (2) is given below) relation (1) gives us

$$\frac{m(2m-1)}{3} < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} < m + \frac{m(2m-1)}{3}.$$

Multiply this relation by $\pi^2/(4m^2)$ and let $m \to \infty$ to obtain

$$\lim_{m\to\infty} \sum_{k=1}^{m} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Proof of (2). By equating imaginary parts in the formula

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n = \sin^n\theta(\cot\theta + i)^n$$