Abstract

There are well defined notions of trace and discriminant for free algebras over a commutative ring. In the case of field extensions, the trace reads if the extension is separable. What we are trying to do is to generalize these concepts in a more general setting, namely that of group schemes actions on schemes. The aim is to develop a general theory for the trace and the discriminant, looking at how classical properties are restated in this wider context and if they provide us with information on actions, for example on their freedom or faithfulness. We also hope that this theory will help us studying the actions of infinitesimal unipotent group schemes on $k(t_1, \ldots, t_n)$.

1. The classical case

Let R be a commutative ring, A/R be a free R-algebra of rank n and $\{x_1, \ldots, x_n\}$ be a basis. The *trace* and *norm* of A/R are respectively defined as

$$Tr_{A/R} : A \to R, \ x \mapsto \operatorname{tr}(m_x) \quad \text{and} \quad N_{A/R} : A \to R, \ x \mapsto \det(m_x)$$

where m_x is the matrix of multiplication by x. The *discriminant* of A/R is the ideal

$$\mathfrak{d}_{A/R} = \det\left(Tr_{A/R}(x_i x_j)\right)_{i,j} R.$$

Example

Let L/K be a finite Galois extension with Galois group G. For any $x \in L$ it holds

$$Tr_{L/K}(x) = \sum_{g \in G} g(x).$$

Transitivity properties

Let A be an integral domain, $F = \operatorname{Frac}(A)$, $F \subseteq K \subseteq L$ be finite and separable field extensions, B be the integral closure of A in K and C be the integral closure of A in L and suppose that B/A is free and C/B is free. Then

$$Tr_{L/F} = Tr_{K/F} \circ Tr_{L/K} \quad \text{and} \quad \mathfrak{d}_{C/A} = \mathfrak{d}_{B/A}^{rk_B(C)} N_{B/A}(\mathfrak{d}_{C/B}),$$

where $N_{B/A}(dB) = N_{B/A}(d)A$.

2. A bit of context

While in the classical case one considers abstract groups and their actions, the objects we will be looking at are group schemes. An affine group scheme $G = \operatorname{Spec}(A)$ over a ring R is completely determined by the R-Hopf algebra A. The group structure of G is given by the maps

$$\Delta : A \to A \otimes_R A, \ i : A \to A, \ \varepsilon : A \to R$$

of comultiplication, coinverse and counit. Abstract groups are the constant objects in the

4. Definition of trace and discriminant

Hereon we work locally, so that we can suppose everything free. In the above context, let $R = B^G$ (the points fixed by the G-action) and α be a generator of $\operatorname{Ann}(I_{A^\vee})$. We define the *trace* of the action of G on X as the R-linear map

$$\operatorname{tr}_{G,\alpha} = v(\alpha) : B \xrightarrow{\rho} B \xrightarrow{\operatorname{id} \otimes \alpha} B \otimes_R R \simeq B.$$

One has in fact that $\operatorname{tr}_{G,\alpha} : B \to B^G$.

Example: recovering the classical case

Let L/K be a finite Galois extension with Galois group G. The action of G on L corresponds to the morphism of K-algebras

$$v : K[G] \to \operatorname{Hom}_K(L, L), \ g \mapsto (x \mapsto g(x)).$$

The augmentation ideal of the Hopf algebra $K[G]$ and its annihilator are respectively

$$I = \ker(\varepsilon : K[G] \to K, g \mapsto 1) = \langle g - h \rangle_{g,h \in G} \quad \text{and} \quad \operatorname{Ann}(I) = \left\langle \sum_{g \in G} g \right\rangle.$$

Then for every $x \in L$ its trace is $\operatorname{tr}_{G,\alpha}(x) = v\left(\sum_{g \in G} g\right)(x) = \sum_{g \in G} g(x) = Tr_{L/K}(x)$.

Let $\{x_1, \ldots, x_n\}$ be a basis of B/R. We define the *discriminant* of the action to be the ideal

$$\mathfrak{d}_G(B) = \det\left(\operatorname{tr}_\alpha(x_i x_j)\right)_{i,j} R.$$

These definitions were given by Childs and Hurley respectively in [CH86] and [Chi87].

5. Transitivity property

Let $H = \operatorname{Spec}(A/J) \subseteq G$ be a finite locally free normal subgroup, $G/H = \operatorname{Spec}(A^H)$. There are naturally induced actions of H on $\operatorname{Spec}(B)$ and of G/H on $\operatorname{Spec}(B^H)$. Suppose $\operatorname{Ann}(I_{(A^H)^\vee}) = \langle \alpha \rangle$, $\operatorname{Ann}(I_{(A/J)^\vee}) = \langle \beta \rangle$, so that the traces for these actions are

$$\operatorname{tr}_{H,\beta} : B \to B^H \quad \text{and} \quad \operatorname{tr}_{G/H,\alpha} : B^H \to B^G.$$

We proved the following: