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Ford Circles Strike Gold

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Abstract. Call a reduced fraction with denominator q a *bronze, silver*, or *gold* approximation with respect to a given irrational number ω if they differ by less than the reciprocal of the product of the square of q and 1, 2, or $\sqrt{5}$, respectively. Suppose A and B are neighboring Farey fractions sandwiching ω, where A's denominator is at least as large as B's denominator. Then B is bronze; either A or B is silver; and at least one but not all of A, B, and their mediant is gold. We prove afresh these results using the geometry of Ford circles rather than a purely algebraic approach, and explore some open questions with respect to the bronze-silver-gold classification while panning for gold fractions among the continued fraction convergents for ω.

1. INTRODUCTION. Given a positive irrational number ω and a reduced fraction $\frac{p}{q}$ of nonnegative integers, we say that $\frac{p}{q}$ is a *bronze, silver*, or *gold* approximation for ω if ω and $\frac{p}{q}$ differ by less than $\frac{1}{q^2}$, $\frac{1}{2q^2}$, or $\frac{1}{\sqrt{5}q^2}$, respectively, terminology introduced in [15, pp. 244–245], [16]. In 1842, Gustav Lejeune Dirichlet showed in [2]—using what is arguably the first application of the *pigeon-hole principle* although he called it the *Schubfachprinzip* or the *drawer principle* [13]—that there are an infinite number of bronze fractions for each ω. Fifty years later, Adolf Hurwitz showed that there are an infinite number of gold fractions for ω [6]. He also showed that for any real number ϵ, $0 < \epsilon < 1$, there exist irrational numbers ω, such as the golden mean $\omega = \phi = \frac{1+\sqrt{5}}{2}$, for which there are only a finite number of fractions $\frac{p}{q}$ where $|\omega - \frac{p}{q}| < \frac{\epsilon}{\sqrt{5}q^2}$; thus there is no need for, say, platinum fraction status. In 1938, Lester Ford simplified Hurwitz's argument using the geometry of what are now called Ford circles [4]. In this article we simplify and extend Ford's argument. In particular, suppose that $A = \frac{a}{b}$ and $B = \frac{c}{d}$ are reduced nonnegative fractions where $b \geq d$, $|ad - bc| = 1$, and ω lies between A and B (which is equivalent to saying that A and B are neighboring Farey fractions sandwiching ω). Then B is bronze; either A or B is silver; and at least one but not all of A, B, and $A \oplus B = \frac{a+c}{b+d}$—called the *mediant* of A and B—is gold. Finally, we explore panning for gold fractions among the continued fraction convergents for ω.

Before doing so, we define a few terms.

2. FAREY SEQUENCES AND FORD CIRCLES. For the remainder of this article unless specified otherwise, when we write a fraction such as $\frac{a}{b}$ we mean that a and b are relatively prime integers. If we identify the fraction $\frac{p}{q}$ with the vector $\begin{bmatrix} p \\ q \end{bmatrix}$, then the mediant $\frac{a}{b} \oplus \frac{c}{d}$ is the fraction equivalent to the vector $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$. Let n be a positive integer. The *Farey sequence* of order n, denoted $\mathcal{F}_n$—named after geologist John Farey (1766–1826)—is that set, in ascending order, of all (reduced) fractions in $[0, 1]$ whose denominators are at most n. Furthermore, fractions $\frac{a}{b}$ and $\frac{c}{d}$ are *adjacent* fractions or *neighbors* if they are adjacent fractions in some Farey sequence $\mathcal{F}_n$. We say that $\frac{c}{d}$ is

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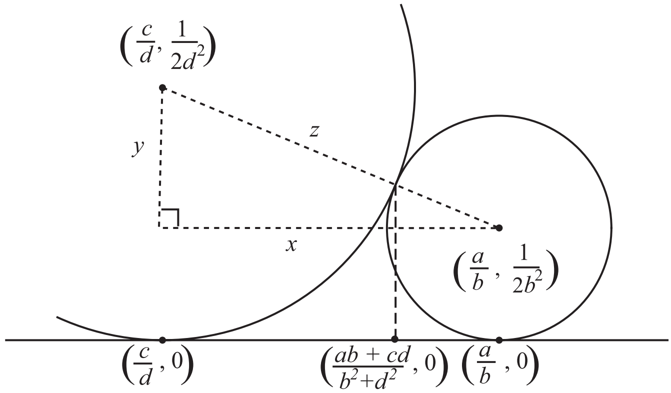

Figure 1. Ford circles $\mathcal{O}_A$ and $\mathcal{O}_B$ for Farey neighbors $A = \frac{a}{b}$ and $B = \frac{c}{d}$.

simpler than $\frac{a}{b}$ if $d < b$. As shown in, for example, [**9**, p. 257] or [**15**, pp. 113–114], $\frac{a}{b}$ and $\frac{c}{d}$ are neighbors if and only if $|ad - bc| = 1$; and two Farey neighbors share the same denominator d only when $d = 1$. The first few Farey series are

$$\mathcal{F}_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}, \ \mathcal{F}_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}, \ \mathcal{F}_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}, \ \mathcal{F}_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}.$$

Furthermore, these Farey properties hold for the set of fractions between any two consecutive integers.

To generate $\mathcal{F}_{n+1}$ from $\mathcal{F}_n$, for each pair of adjacent fractions $A = \frac{a}{b}$ and $B = \frac{c}{d}$ in $\mathcal{F}_n$ where $b + d = n + 1$, insert their mediant between them. As shown in [**9**, p. 257] or [**15**, pp. 113–114], the mediant $A \oplus B$ is already in reduced form and lies between A and B, and is a Farey neighbor of both A and B; and $A \oplus B$ is the simplest fraction in the open interval from A to B. Let n be a nonnegative integer. As follows by induction, the *iterated mediant* of Farey neighbors A and B, $nA \oplus B = \frac{na+c}{nb+d}$, is a Farey neighbor of both A and $(n-1)A \oplus B$ for all $n \geq 1$. Since the mediant operator is commutative, we have $nA \oplus B = B \oplus nA$. With respect to identifying fractions with vectors, we have $nA \oplus B \equiv n\mathbf{A} + \mathbf{B}$ where $\mathbf{A}$ and $\mathbf{B}$ are vectors equivalent to A and B. Observe that the iterated mediant $nA \oplus B$ converges monotonically to A as $n \to \infty$.

The *Ford circle* for the fraction $A = \frac{a}{b}$, denoted $\mathcal{O}_A$, is the circle with center $(\frac{a}{b}, \frac{1}{2b^2})$ and radius $\frac{1}{2b^2}$, as shown in Figure 1. We denote the radius of $\mathcal{O}_A$ as $|\mathcal{O}_A| = \frac{1}{2b^2}$.

Figure 2 shows the relationships between Ford circles and the iterated mediant of two Farey neighbors A and B. As n increases positively in Figure 2, the Ford circles for $nA \oplus B$ form a sequence of shrinking disks converging monotonically to the point $(A, 0)$. Similarly, the Ford circles for $A \oplus nB$ converge to the point $(B, 0)$.

To give geometric interpretations to bronze, silver, and gold, we have that a fraction $\frac{a}{b}$ is silver for ω if the vertical line L with equation $x = \omega$ intersects the interior of the Ford circle for $\frac{a}{b}$. Analogously, the fraction $\frac{a}{b}$ is bronze or gold if L intersects the interiors of the Ford circles concentrically rescaled by factors of 2 or $\frac{2}{\sqrt{5}} \approx 0.894$, respectively. For example, Figure 3 illustrates this geometric interpretation, with $\frac{106}{39}$ bronze for e, $\frac{87}{32}$ silver, and $\frac{193}{71}$ gold, where the solid circles are Ford circles, and the dashed circles are Ford circles scaled by $2/\sqrt{5}$.

This next proposition demonstrates a striking relationship between Farey neighbors and their Ford circles, [**4**, pp. 588, 592], whose proof we include for completeness.

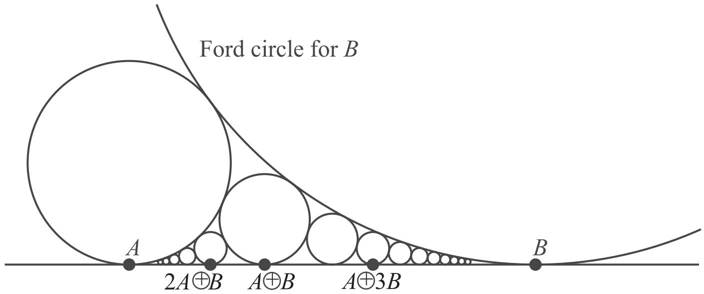

Figure 2. The iterated mediants $nA \oplus B$ and $A \oplus nB$ with respect to Ford circles, $A = \frac{3}{2}$ and $B = \frac{2}{1}$.

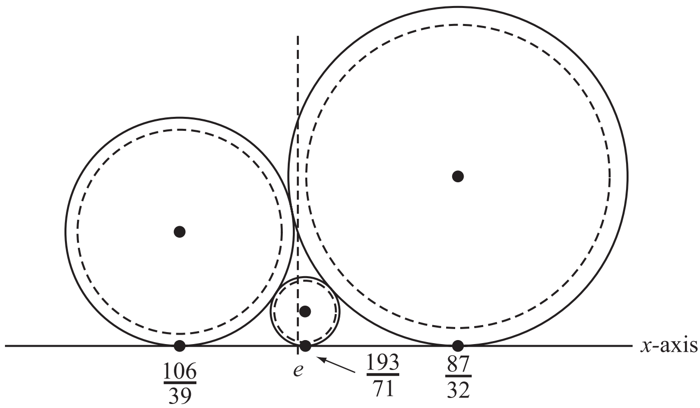

Figure 3. Neighboring bronze, silver, and gold fractions for $e \approx 2.718$.

Proposition 1 (Ford circles for Farey neighbors). *Let $A = \frac{a}{b}$ and $B = \frac{c}{d}$ be non-negative fractions. A and B are Farey neighbors if and only if their Ford circles are tangent.*

Proof. Let r and s be the radii of the two circles in Figure 1. The two circles will be tangent exactly when the hypotenuse of the right triangle has length $r + s$. The tangency condition is thus

$$(r - s)^2 + \left(\frac{a}{b} - \frac{c}{d}\right)^2 = (r + s)^2.$$

This condition simplifies to

$$\left(\frac{ad - bc}{bd}\right)^2 = 4rs.$$

Substituting $r = \frac{1}{2d^2}$ and $s = \frac{1}{2b^2}$ converts this equation to the equation $(ad - bc)^2 = 1$. Therefore the circles are tangent if and only if $|ad - bc| = 1$. ∎

Corollary 2 (Intersection of Ford circles). *The intersection of any two distinct Ford circles is either empty or a single shared boundary point.*

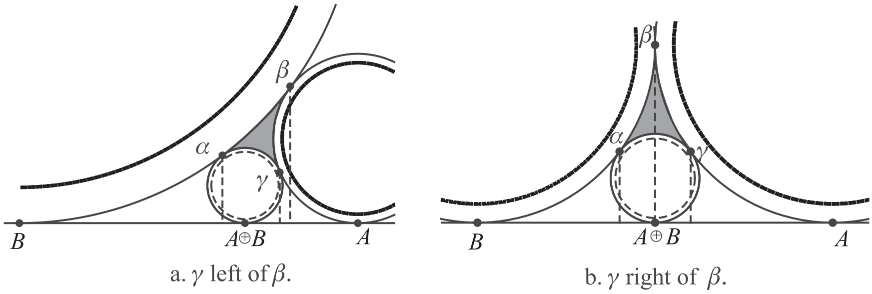

Figure 4. Mesh triangles for Farey neighbors and their mediant.

Proof. The proof is by induction on the recursive way in which the Farey series $\mathcal{F}_n$ is defined. Observe that the Farey circles for $\mathcal{F}_1$ are tangent circles with centers at $(0,\ \frac{1}{2})$ and $(1,\ \frac{1}{2})$. The Farey circles for $\mathcal{F}_2$ include the circle with center $(\frac{1}{2},\ \frac{1}{8})$ whose intersection with each circle of $\mathcal{F}_1$ is a single point, and so on. ■

Figure 3 illustrates the next corollary where $A = \frac{106}{39}$ and $B = \frac{87}{32}$ are neighboring Farey fractions that sandwich $\omega = e$. In this case B is at least bronze (and is in fact silver).

Corollary 3 (A bronze and silver sandwich). *Suppose the irrational number ω lies between Farey neighbors $A = \frac{a}{b}$ and $B = \frac{c}{d}$ with $b \geq d$. Then B is bronze, and at least one of A and B is silver with respect to ω.*

Proof. Since $|A - B| = \frac{1}{bd}$, the projection of the concentrically-rescaled-by-a-factor-of-2 Ford circle for B onto the x-axis includes the interval from A to B, as does the projection of the union of the interiors of the Ford circles for A and B with the exception of the midpoint of A and B when they are adjacent integers. Note that this midpoint is rational so it cannot equal ω. ■

3. GOLD AMONG FAREY NEIGHBORS AND THEIR MEDIANT. To show Hurwitz's result that there are an infinite number of gold fractions for any irrational number, Ford defined the *mesh triangle* associated with Farey neighbors A and B as the region in the plane between the Ford circles for A, B, and $A \oplus B$, as shown in Figure 4 where α, β, and γ are the points of tangency between the circles. Ford demonstrated that if the vertical line through $(\omega,\ 0)$ intersects the mesh triangle associated with A and B, then at least one of A, B, and $A \oplus B$ is a gold fraction for ω [**4**, Theorem 6, p. 592]; thus, since every vertical line through an irrational point on the x-axis intersects an infinite number of mesh triangles, we have Hurwitz's result.

However, we exploit the Ford circle idea to reach a stronger result: that for each pair of Farey fractions A and B sandwiching ω, at least one but not all of A, B, and $A \oplus B$ is gold. That is, instead of restricting ω to lie in the projection of the mesh triangle to obtain gold, ω must simply lie between A and B. To develop terminology for explaining this result, consider three mutually tangent circles C_0, C_t, and C_u that are tangent to the x-axis at the real numbers 0, t, and u with $0 < u < t$, as depicted in Figure 5. Let C_0', C_t', and C_u' be concentrically shrunk copies of C_0, C_t, and C_u with shrinking factor $\frac{2}{\sqrt{5}}$. These are shown as dashed circles in the figure. We may rescale the whole figure so that C_0 has radius 1. By symmetry we may assume C_t is no larger than C_0, so $t \leq 2$. We refer to $(C_0',\ C_t',\ C_u')$ as a *radical triad* (because the three circles are scaled by a radical).

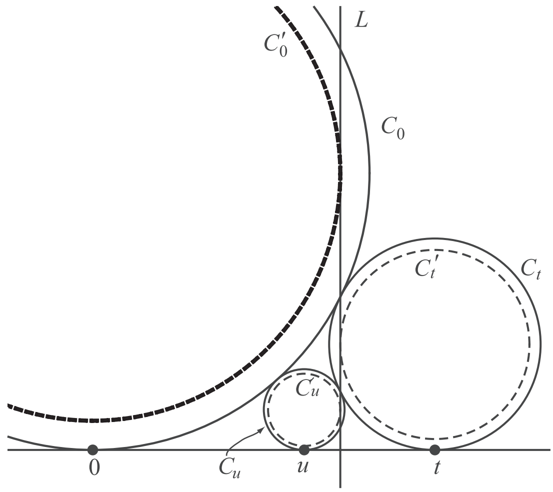

Figure 5. A radical triad (C_0', C_t', C_u'), with each member tangent to vertical line L.

To illustrate how the circles C_0 and C_t are related to the Ford circles for neighboring Farey fractions, consider from Figure 2 the Ford circles for $A = \frac{3}{2}$ and $A \oplus B = \frac{5}{3}$. To find the t value related to this pair of Ford circles, we first translate the two Ford circles to the left by A so that they now rest on the x-axis at 0 and $\frac{1}{6}$. Scale this arrangement of circles by 8, the reciprocal of the radius of $\mathcal{O}_A$. The two transformed circles are now C_0 and C_t where $t = (8) \cdot (\frac{1}{6}) = \frac{4}{3}$ with respect to the original number line. To generate C_0 and C_t corresponding to the Ford circles for Farey fractions $F_1 < F_2$ when F_1's denominator is more than F_2's denominator, we would first reflect their Ford circles about the y-axis, and continue as we did with the two Ford circles in this example.

The next two propositions highlight properties of radical triads.

Proposition 4 (A unique tangent for a radical triad). *There is a unique value of t, namely $t = \sqrt{5} - 1$, such that each of the circles in (C_0', C_t', C_u') is tangent to the same vertical line L, with C_0' and C_u' to the left of L and C_t' to the right.*

Proof. Let r be the radius of C_t and s the radius of C_u. First we find equations expressing the tangencies between the three circles C_0, C_t, and C_u. For C_0 and C_t, consider the right triangle—much like the dashed triangle in Figure 1—whose hypotenuse is the line segment joining the centers of C_0 and C_t and whose other two sides are vertical and horizontal, with a vertical side at the center of C_0 and a horizontal side at the center of C_t. By the Pythagorean theorem, we have $t^2 + (1 - r)^2 = (1 + r)^2$, which simplifies to $t^2 = 4r$, or $r = t^2/4$. Similarly, the tangency of C_0 and C_u gives $s = u^2/4$; and the tangency of C_t and C_u gives $(t - u)^2 + (r - s)^2 = (r + s)^2$, or $(t - u)^2 = 4rs$. With $r = t^2/4$ and $s = u^2/4$, the equation $(t - u)^2 = 4rs$ becomes $(t - u)^2 = t^2u^2/4$.

Next we determine the value of t for which the vertical line L tangent to the right side of C_0' is also tangent to the left side of C_t'. Tangency to the right side of C_0' means that the x-coordinate of L is $\frac{2}{\sqrt{5}}$ and tangency to the left side of C_t' means the x-coordinate of L is $t - \frac{2}{\sqrt{5}}r = t - \frac{2}{\sqrt{5}} \cdot \frac{t^2}{4}$. Equating these two x-coordinates leads to the equation $t^2 - 2\sqrt{5}\,t + 4 = 0$ with roots $t = \sqrt{5} \pm 1$. We want the root with $t < 2$, namely, $t = \sqrt{5} - 1$. Similarly, for L to be tangent to the right side of C_u' means that $\frac{2}{\sqrt{5}} = u + \frac{2}{\sqrt{5}} \cdot \frac{u^2}{4}$ which simplifies to $u^2 + 2\sqrt{5}\,u - 4 = 0$ with roots $u = \pm 3 - \sqrt{5}$. We want the positive root, $u = 3 - \sqrt{5}$.

Finally, to see that C_t and C_u are tangent when $t = \sqrt{5} - 1$ and $u = 3 - \sqrt{5}$, the condition for tangency is $(t - u)^2 = t^2u^2/4$. Note that $(t - u)^2 = (2\sqrt{5} - 4)^2 = 36 - 16\sqrt{5}$, and that $t^2u^2/4 = (6 - 2\sqrt{5})(14 - 6\sqrt{5})/4 = (3 - \sqrt{5})(7 - 3\sqrt{5}) = 36 - 16\sqrt{5}$. ■

We note the following monotonicity property: As t varies over the interval $(0, 2]$ the left edge of C_t' varies monotonically with t, moving to the left as t decreases and to the right as t increases. This is because the left edge of C_t' has x-coordinate $t - \frac{2}{\sqrt{5}}r = t - \frac{1}{2\sqrt{5}}t^2$ and the graph of this quadratic function of t is an inverted parabola crossing the horizontal axis at $t = 0$ and $t = 2\sqrt{5}$, so the parabola increases on the t-interval $[0, \sqrt{5}]$, which contains the interval $[0, 2]$.

Proposition 5 (Verticals piercing a radical triad). *As t varies over the interval $(0, 2]$, each vertical line $x = \omega$ with $0 \leq \omega \leq t$ intersects the interior of either one or two but not all of the circles C_0', C_t', and C_u', except when $t = \sqrt{5} - 1$ and the line $x = \omega$ is the line L tangent to all three circles.*

Proof. Consider first what happens as t decreases from $t = \sqrt{5} - 1$. By the monotonicity property, the left edge of C_t' is always to the left of the right edge of C_0'. The same principle shows that the right edge of C_u' is always to the left of the left edge of C_t', because we may rescale and translate the figure so that C_t remains at a fixed size and position while C_u varies relative to C_t, ever staying tangent to it. That is, imagining that C_t remains fixed in size and position as t decreases, we see that C_0 swells in size, rolling to the left along the x-axis and clockwise along C_t; in order for C_u to remain in tangent relation to both of the other circles and the x-axis, it also will swell in size and roll to the left and clockwise along C_t.

Thus for $t < \sqrt{5} - 1$, the left edge of C_t' lies strictly between the right edge of C_u' and the right edge of C_0'. Hence, for $t < \sqrt{5} - 1$, every vertical line $x = \omega$ with $0 \leq \omega \leq t$ intersects the interior of either one or two but not all of C_0', C_t', and C_u'. Figure 4a illustrates this case.

Now consider what happens as t increases over the interval $[\sqrt{5} - 1, 2]$. Applying the monotonicity property twice as above, we see that the left edge of C_t' is always between the right edge of C_0' and the right edge of C_u'. (This time, when fixing C_t in size and position as t increases, the other two circles shrink, rolling to the right.) Moreover, the left edge of C_u' remains to the left of the right edge of C_0' since this is true when $t = 2$ and $u = 1$, as illustrated in Figure 4b. Thus the desired conclusion also holds for t in the interval $[\sqrt{5} - 1, 2]$. ■

Note that if the radical triad (C_0', C_t', C_u') had been defined using a shrinking factor less than $\frac{2}{\sqrt{5}}$, then for t values near $\sqrt{5} - 1$ there would be vertical lines $x = \omega$ near L that do not intersect any circle of the triad.

Our main result follows. For an algebraic proof using the number line—rather than our geometric proof below using Ford circles in the plane—see [10, Lemma 1.8].

Corollary 6 (At least one but not all is gold). *If Farey neighbors A and B sandwich the irrational number ω, then at least one but not all of A, B, and $A \oplus B$ is gold.*

Proof. Let $A = \frac{a}{b}$ and $B = \frac{c}{d}$ with $b \leq d$ so that $\mathcal{O}_A$ is at least as large as $\mathcal{O}_B$. By reflection symmetry, we may assume $A < B$. By horizontal translation and rescaling of the plane, we may convert $\mathcal{O}_A$, $\mathcal{O}_B$, and $\mathcal{O}_{A\oplus B}$ into the circles C_0, C_t, and C_u in Proposition 5, respectively. Therefore the proposition gives the desired result. ■

4. GOLD CONVERGENTS FOR SIMPLE CONTINUED FRACTIONS.

To illustrate how to generate Farey neighbors sandwiching the irrational number ω, we consider the *simple continued fraction* algorithm for ω, also called the *regular* continued fraction algorithm. This algorithm corresponds to jumping successively along a sequence of bronze, silver, and gold fractions for ω from one fraction—now called the *convergent* of the continued fraction for ω—to another fraction. From a long tradition, we say that the list of *partial denominators* $[m_0;\ m_1,\ m_2,\ m_3,\ \ldots]$ denotes a continued fraction where m_0 is a nonnegative integer and m_i are positive integers for all positive integers i. Each list of the first $i+1$ partial denominators of the continued fraction evaluates to convergent i, denoted C_i, for all $i \geq 0$:

$$C_0 = m_0, \quad C_1 = m_0 + \frac{1}{m_1}, \quad C_2 = m_0 + \frac{1}{m_1 + \frac{1}{m_2}}, \ \ldots .$$

When the sequence C_i converges to a number ω, we write $\omega = [m_0;\ m_1,\ m_2,\ \ldots]$. Similarly, we write $C_i = [m_0;\ m_1,\ m_2,\ \ldots, m_i]$. The initial partial denominator m_0 is $m_0 = \lfloor \omega \rfloor$. For each $i \geq 1$, m_i is defined by $\omega = [m_0;\ m_1,\ m_2,\ \ldots,\ m_{i-1} + r]$ for some positive real number r less than 1, and

$$m_i = \left\lfloor \frac{1}{r} \right\rfloor, \tag{1}$$

which means that $m_i \geq 1$.

To proceed directly from one convergent to the next we use a Fibonacci-like recursion among the previously defined convergents. The symbols $C_{-1} = \frac{1}{0}$ and $C_{-2} = \frac{0}{1}$ are called *preconvergents*. The following proposition is a list of simple continued fraction properties. Proofs of the first five items can be found in almost any elementary number theory text such as [**14**, Theorem 12.9, pp. 485–486] or [**15**, Proposition 16, pp. 318–319]. We illustrate property (ii) in Example 9. The award-winning article by Richards [**12**] is an algorithm based on items (iv) and (v): to find the next convergent for ω following the convergents $B = \frac{c}{d}$ and $A = \frac{a}{b}$ that sandwich ω where $b > d$, generate the sequence $A \oplus B$, $2A \oplus B$, and so on until the fraction $nA \oplus B$ is on the same side of ω as A for some integer n; the next convergent is taken to be $(n-1)A \oplus B$; the reason this algorithm works is because the real number s defined in property (iv) exceeds 1. Standard algebraic proofs of property (vi) appear in Hardy and Wright's classic number theory text [**5**, Theorem 184, p. 184], and [**11**, Satz 2.11]. For completeness we include proofs for properties (v) and (vi).

In the spirit of Lester Ford, we offer a Ford circle proof for property (vi)—which may be new in the literature.

Proposition 7 (Convergents of a continued fraction). *The simple continued fraction for the positive irrational number ω has the following properties.*

i. *ω's partial denominators form a unique sequence, $\omega = [m_0;\ m_1,\ m_2,\ \ldots]$.*
ii. *The convergents C_k for ω, for each integer $k \geq 0$, are defined recursively from ω's*

partial denominators by $C_k = m_k C_{k-1} \oplus C_{k-2}$ where

$$\begin{array}{ll} p_{-2} = 0, & q_{-2} = 1, \\ p_{-1} = 1, & q_{-1} = 0, \\ p_k = m_k p_{k-1} + p_{k-2}, & q_k = m_k q_{k-1} + q_{k-2}. \end{array} \tag{2}$$

iii. *C_{k-1} and C_k are adjacent fractions in some Farey sequence for all $k \geq 1$.*
iv. *The partial denominators m_k are determined by solving $\omega = sC_{k-1} \oplus C_{k-2}$ and taking $m_k = \lfloor s \rfloor$ for all $k \geq 0$, where $s > 1$ is a real number. In fact, $s = \frac{1}{r}$, where r is defined in (1).*
v. *The sequence C_k oscillates about ω, for all $k \geq 0$.*
vi. *The sequence C_k includes every silver and gold fraction for ω.*

Proof. For (v), we know that $m_0 = \lfloor \omega \rfloor$, so $C_0 = \lfloor \omega \rfloor$. Solving $\omega = sC_0 \oplus C_{-1} = \frac{s\lfloor \omega \rfloor + 1}{s \cdot 1 + 0}$ for s gives $s = \frac{1}{\omega - \lfloor \omega \rfloor} > 1$. The function $sC_0 \oplus C_{-1}$ converges monotonically to C_0 as s increases from 0. With $m_1 = \lfloor s \rfloor$, we have $(m_1 + 1)C_0 \oplus C_{-1}$ and $C_1 = m_1 C_0 \oplus C_{-1}$ are on opposite sides of ω, which means that C_0 and C_1 are on opposite sides of ω. Assume that C_{i-1} and C_{i-2} are on opposite sides of ω for all i, where $2 \leq i \leq k$ for some positive integer $k \geq 2$. Since $sC_{k-1} \oplus C_{k-2}$ monotonically converges to C_{k-1} as s increases from 0, we see that C_k and C_{k-2} are on the same side of ω, which means that C_k and C_{k-1} are on opposite sides of ω.

For (vi), suppose that $Q = \frac{p}{q}$ is silver or gold but not a convergent of ω. Because the denominators of ω's convergents form a strictly increasing sequence of positive integers after the initial convergent C_0, there exist successive convergents $B = \frac{c}{d}$ and $A = \frac{a}{b}$ with $d \leq q \leq b$. This restriction means that the sizes (radii) of the Ford circles $\mathcal{O}_A$, $\mathcal{O}_B$, and $\mathcal{O}_Q$ satisfy $|\mathcal{O}_A| \leq |\mathcal{O}_Q| \leq |\mathcal{O}_B|$. By symmetry, we may assume without loss of generality that $A < B$. By (v), we know that $A < \omega < B$. The circles $\mathcal{O}_A$ and $\mathcal{O}_B$ are tangent since A and B are successive convergents. See Figure 6. Since $|\mathcal{O}_Q| \geq |\mathcal{O}_A|$, the circle $\mathcal{O}_Q$ cannot fit in the shaded region between $\mathcal{O}_A$, $\mathcal{O}_B$, and the x-axis; hence we cannot have $A < Q < B$.

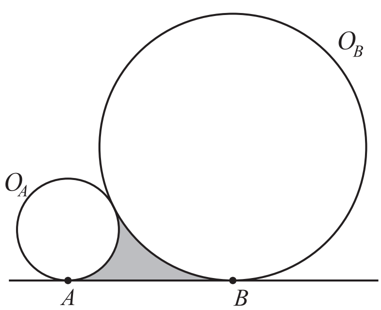

Figure 6. Not enough room between $\mathcal{O}_A$ and $\mathcal{O}_B$.

If $Q > B$, then $\mathcal{O}_Q$ lies to the right of the line $x = B$ because $|\mathcal{O}_Q| \leq |\mathcal{O}_B|$. Hence the line $x = \omega$ fails to intersect $\mathcal{O}_Q$, contradicting the assumption that Q is silver or gold for ω. Thus we must have $Q < A$.

Assuming $Q < A$, let $\mathcal{O}'_B$ be a copy of $\mathcal{O}_B$ reflected across the vertical line $x = A$. We distinguish three cases according to whether $\mathcal{O}'_B$ is disjoint from $\mathcal{O}_B$, is tangent to $\mathcal{O}_B$, or intersects $\mathcal{O}_B$ in two points, as illustrated in Figure 7.

In the first case, let P be the path, shown in bold in Figure 7a, that starts on the x-axis, moves vertically upward to the leftmost point of $\mathcal{O}_A$, then moves upward along

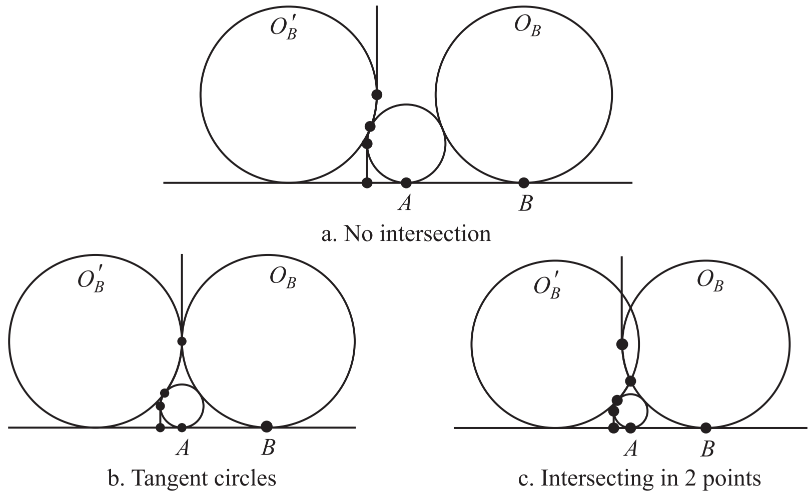

Figure 7. The Ford circle $\mathcal{O}_B$ and its clone $\mathcal{O}_B'$.

an arc of $\mathcal{O}_A$ to the point of tangency between $\mathcal{O}_A$ and $\mathcal{O}_B'$, then upward along an arc of $\mathcal{O}_B'$ to the rightmost point of $\mathcal{O}_B'$, then vertically upward from there. Since $|\mathcal{O}_A| \leq |\mathcal{O}_Q| \leq |\mathcal{O}_B|$, the circle $\mathcal{O}_Q$ lies in the region formed by P and all the points to the left of P. This region does not extend to the right of the vertical line $x = A$, so the same is true of $\mathcal{O}_Q$. Thus, the line $x = \omega$ is disjoint from $\mathcal{O}_Q$, so Q cannot be silver or gold for ω. Exactly the same argument applies in the case when $\mathcal{O}_B'$ is tangent to $\mathcal{O}_B$.

If $\mathcal{O}_B'$ intersects $\mathcal{O}_B$ in two points, we modify the definition of the path P so that it first proceeds upward to the leftmost point of $\mathcal{O}_A$, then moves along $\mathcal{O}_A$ to the tangency point with $\mathcal{O}_B'$, then along $\mathcal{O}_B'$ to its lower intersection point with $\mathcal{O}_B$, then along $\mathcal{O}_B$ to the leftmost point of $\mathcal{O}_B$, then vertically upward. Again $\mathcal{O}_Q$ lies entirely within the region to the left of P inclusive of P itself. See Figure 7c. Hence, $\mathcal{O}_Q$ does not extend to the right of the line $x = A$ that passes through the intersection points of $\mathcal{O}_B$ and $\mathcal{O}_B'$, and we have the same contradiction as before, and the proof for (vi) is complete. ■

Observe that item (vi) of the proposition is best possible in the sense that some irrationals ω have bronze approximations that fail to be convergents in the continued fraction for ω. For example, suppose the continued fraction for ω begins $[0;\ 1,\ 2,\ m_3,\ m_4,\ \ldots]$; so ω lies between $\frac{2}{3}$ and $\frac{3}{4}$, and its first few convergents are $0, 1, \frac{2}{3}$. The fraction $\frac{1}{2}$ is then bronze for ω but fails to be a convergent.

5. PANNING FOR GOLD. To pan for gold fractions in a veritable river-like sequence of convergents for any particular ω, we first use our preceding work, gathering several prospector tools as the following proposition. Fortunately we may pan with confidence knowing that *there's gold in them thar hills*—a phrase made famous by the cartoon character Yosemite Sam—because every existing gold fraction is a convergent by Proposition 7(vi). Algebraic proofs for statements 1, 2, and 3 appear in [**5**, Theorems 171, 183, 195]; and proofs for statements 2 and 3 also appear in [**11**, Satz 2.14, 2.15].

Proposition 8 (Bronze, Silver, and Gold convergents). *Let C_i be the regular convergents for ω. For every $i \geq 0$,*

1. *C_i is bronze for every $i \geq 0$.*

2. *Either C_i or C_{i+1} is silver.*
3. *At least one of each triple $C_i,\ C_{i+1},\ C_{i+2}$ is gold.*
4. *If $m_{i+1} \geq 2$, then C_i is silver.*
5. *If $m_{i+1} \geq 3$, then C_i is gold.*
6. *If neither C_i nor C_{i+1} is gold, then C_{i+2} is gold and both m_{i+1} and m_{i+2} are* 1.

Proof. Let $C_i = m_i C_{i-1} \oplus C_{i-2} = \frac{c}{d}$ and $B = (1 + m_i)C_{i-1} \oplus C_{i-2} = \frac{a}{b}$, and $A = B \oplus C_i$. We have $B = C_i \oplus C_{k-1}$, and $A = C_i \oplus B = C_i \oplus C_i \oplus C_{i-1} = 2C_i \oplus C_{i-1}$. For statements 1 and 2, by Proposition 7(iii) and 7(v), we know that C_i and C_{i+1} are Farey neighbors sandwiching ω and that the denominator of C_i is no more than that of C_{i+1}. The desired results follow by Corollary 3. For statement 3, we know that at least one of C_i, C_{i+1}, and $C_i \oplus C_{i+1}$ is gold by Corollary 6. If neither C_i nor C_{i+1} is gold then $C_i \oplus C_{i+1}$ is gold, which means it is a convergent of ω by property (vi); and by property (ii) it must be convergent C_{i+2}. For statement 4, suppose that C_i fails to be silver. By properties 2 and (vi), we know that B must be silver and that B is a convergent of ω, which means that $B = C_{i+1}$, making $m_{i+1} = 1$, a contradiction. For statement 5, suppose that C_i fails to be gold. By property 3 either B or A is gold, which again means that m_{i+1} is either 1 or 2, a contradiction. For statement 6, suppose that neither C_i nor C_{i+1} is gold. However, we know that either B or A is gold. If B is gold, then $B = C_{i+1}$, a contradiction. So A must be gold. The fraction A cannot be C_{i+1}. Because $A = 2C_i \oplus C_{i-1} = B \oplus C_i$, we have $B = C_{i+1}$ and $A = C_{i+2}$, making $m_{i+1} = 1 = m_{i+2}$. ■

Observe that properties 4 and 5 of the proposition are in accord with the well-known observation that truncating an infinite continued fraction just before a large partial denominator gives a particularly good rational approximation. For example, a gold convergent for π from Example 10 below is $\frac{355}{113} = [3;\ 7,\ 15,\ 1]$. Furthermore, as an inverse-like statement for properties 4 and 5, Corollary 6 implies that if a partial denominator m_{i+1} is 1, then the three consecutive convergents $C_{i-1},\ C_i$, and C_{i+1} cannot all be gold.

Example 9 (Finding natural gold). With $\omega = e$, as Leonhard Euler discovered in 1744 [**1**, **3**], we have $e = [2;\ 1,\ 2,\ 1,\ 1,\ 4,\ 1,\ 1,\ 6,\ \ldots]$. To generate these partial denominators, we start with m_0. Solving $e = sC_{-1} \oplus C_{-2}$ gives $s = e$, and $m_0 = \lfloor s \rfloor = 2$, making $C_0 = \frac{2}{1}$. Solving $e = sC_0 \oplus C_{-1}$ gives $s \approx 1.392$, $m_1 = 1$, and $C_1 = \frac{3}{1}$. And so on. In Table 1, results are given up through $k = 10$. The last column gives the assay report of convergent C_k. For example, with $k = 5$, the last column is S, which means $\frac{87}{32}$ is silver. By statement 5 of Proposition 8, we see that C_4, C_7, and so on are all gold. And a natural conjecture is that the pattern of metals in the convergents of e has period 3, namely, bronze-gold-silver, or *BGS*, starting with $C_0,\ C_1,\ C_2$.

Example 10 (A patternless assay). With $\pi = [3;\ 7,\ 15,\ 1,\ 292,\ 1,\ 1,\ 1,\ 2,\ 1, \ldots]$, the assay report is *GGBGBGBGBGBGG*, whose continuation we conjecture—as perhaps might be expected—is nonperiodic. Does π have a convergent that is but silver?

Example 11 (Finding radical gold). We consider $\sqrt{2} = [1;\ 2,\ 2,\ 2, \ldots]$, $\sqrt{3} = [1;\ 1,\ 2,\ 1,\ 2,\ \ldots]$, $\sqrt{5} = [2;\ 4,\ 4,\ 4,\ \ldots]$, $\sqrt{7} = [2;\ 1,\ 1,\ 1,\ 4,\ 1,\ 1,\ 1,\ 4,\ \ldots]$, and $\sqrt{10} = [3;\ 6,\ 6,\ 6,\ \ldots]$. By property 5 of Proposition 8, the convergents C_i of $\sqrt{5}$ and $\sqrt{10}$ are all gold for all $i \geq 1$. By property 4, all C_i for $\sqrt{2}$ are silver; by calculation, the first six convergents are all gold; so perhaps they are all gold. By

Table 1. Finding gold for e, where $B \equiv$ bronze, $S \equiv$ silver, $G \equiv$ gold.

k	C_k	s	m_k	assay
0	$\frac{2}{1}$	2.718	2	B
1	$\frac{3}{1}$	1.392	1	G
2	$\frac{8}{3}$	2.550	2	S
3	$\frac{11}{4}$	1.819	1	B
4	$\frac{19}{7}$	1.220	1	G
5	$\frac{87}{32}$	4.536	4	S
6	$\frac{106}{39}$	1.867	1	B
7	$\frac{193}{71}$	1.153	1	G
8	$\frac{1264}{465}$	6.528	6	S
9	$\frac{1457}{536}$	1.8945	1	B
10	$\frac{2721}{1001}$	1.117	1	G

properties 1 and 4, all C_i for $\sqrt{3}$ are bronze and C_{2i+1} are all silver; by calculation it appears that C_{2i+1} are all gold and C_{2i} are but bronze.

To demonstrate this $\sqrt{3}$ conjecture, we show that C_{2i} are but bronze, which, by property 3, implies that C_{2i+1} are gold for all $i \geq 0$. We know that $C_0 = 1$ is bronze and $C_1 = 2$ is gold. For $i \geq 0$, our strategy is to show that that $2q_{2i}^2|\sqrt{3} - \frac{p_{2i}}{q_{2i}}| > 1$ where p_i and q_i are defined in (2) of Proposition 7(ii). Observe that $q_0 = 1$; $q_1 = 1$; $q_2 = 2q_1 + q_0 = 3$; $q_3 = q_2 + q_1$; $q_4 = 2q_3 + q_2 = 3q_2 + 2q_1$; and so on. We write these equations as the matrix equations

$$\begin{bmatrix} q_{2i+1} \\ q_{2i+2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^i \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = MD^iM^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

where M and D are the matrix of eigenvectors and the diagonal matrix of eigenvalues of the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$. Similarly, $\begin{bmatrix} p_{2i+1} \\ p_{2i+2} \end{bmatrix} = MD^iM^{-1} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$. Via a CAS, we have $p_{2i+2} = \frac{1}{2\sqrt{3}}((2 - \sqrt{3})^i(-9 + 5\sqrt{3}) + (2 + \sqrt{3})^i(9 + 5\sqrt{3}))$ and $q_{2i+2} = \frac{1}{2\sqrt{3}}((2 - \sqrt{3})^i(-5 + 3\sqrt{3}) + (2 + \sqrt{3})^i(5 + 3\sqrt{3}))$. The left-hand side of our condition for bronze, $2q_{2i+2}^2|\sqrt{3} - C_{2i+2}| > 1$, simplifies as

$$\frac{2}{3}\left(\left(26\sqrt{3} - 45\right)\left(2 - \sqrt{3}\right)^{2i} + \sqrt{3}\right),$$

which decreases monotonically to $\frac{2\sqrt{3}}{3} \approx 1.15 > 1$. Therefore, yes, the convergents of $\sqrt{3}$ forever oscillate between bronze and gold.

Finally, by property 5, the $\sqrt{7}$ convergents C_{4i+3} are all gold; does $\sqrt{7}$ have a pattern that cycles with period 4?

We close with one last example and a general question.

Lemma 12 (Gold in the golden mean). *For* $\phi = \frac{1+\sqrt{5}}{2} = [1;\ 1,\ 1,\ 1,\ \ldots]$, *each* C_{2i+1} *is gold and each* C_{2i+2} *is but silver for all integers* $i \geq 0$.

Proof. The convergents for ϕ are the ratios of successive Fibonacci numbers, f_i: $1,\ 1,\ 2,\ 3,\ 5,\ 8$, and so on, where $f_0 = 1,\ f_1 = 1,\ f_2 = 2$, and the convergents of ϕ are $C_i = \frac{f_{i+1}}{f_i}$: $1,\ 2,\ \frac{3}{2},\ \frac{5}{3}$, and so on. The following assay tool list is a few properties of these convergents, where n is a positive integer. Proofs for each of these statements may be found, for example, in [7]; for completeness, we prove them anew.

a. $C_n > \phi$ when n is odd, and $C_n < \phi$ when n is even.
b. Cassini's identity: $|f_{n+1}^2 - f_n f_{n+2}| = 1$ for all nonnegative integers n.
c. $|\phi - C_n| = \frac{2}{f_n(\sqrt{5}f_n + 2f_{n+1} - f_n)}$ for all $n \geq 1$.
d. $|\phi - C_n| < \frac{1}{2f_n^2}$ for all $n > 1$.
e. When n is odd, $0 < C_n - \phi < \frac{1}{\sqrt{5}f_n^2}$. When n is even, $\phi - C_n > \frac{1}{\sqrt{5}f_n^2}$.

For (a), note that $C_0 = 1 < \phi$; and the result follows by Proposition 7(v). For (b), $|f_1^2 - f_0 f_2| = |1^2 - 1\cdot 2| = 1$. Suppose that $|f_n^2 - f_{n-1}f_{n+1}| = 1$ for some $n > 0$. Since $f_{n+1} = f_n + f_{n-1}$, we have $1 = |f_n^2 - f_{n-1}f_{n+1}| = |f_n^2 - (f_{n+1} - f_n)f_{n+1}|$, which we rewrite as

$$|f_n^2 - f_{n+1}^2 + f_n f_{n+1}| = |f_{n+1}^2 - f_n(f_n + f_{n+1})| = |f_{n+1}^2 - f_n f_{n+2}|.$$

For (c),

$$|\phi - C_n| = \frac{|\sqrt{5}f_n - (2f_{n+1} - f_n)|}{2f_n} = \frac{\frac{1}{2}|5f_n^2 - (4f_{n+1}^2 - 4f_{n+1}f_n + f_n^2)|}{f_n(\sqrt{5}f_n + 2f_{n+1} - f_n)},$$

whose numerator by (b) is

$$2|f_{n+1}^2 - f_{n+1}f_n - f_n^2| = 2|f_{n+1}^2 - f_n(f_{n+1} + f_n)| = 2|f_{n+1}^2 - f_n f_{n+2}| = 2,$$

giving the desired result. For (d), the inequality is true for $n = 2$ and $n = 3$ by direct evaluation. Let $n > 3$. Since $\sqrt{5}f_n - f_n > f_n$ and $f_n + 2f_{n+1} \geq 4f_n$, from (c) we have

$$|\phi - C_n| = \frac{2}{f_n(\sqrt{5}f_n + 2f_{n+1} - f_n)} < \frac{2}{f_n(f_n + 2f_{n+1})} < \frac{1}{2f_n^2}.$$

For (e), by (a) with n odd, $f_{n+1} > f_n\phi$ is equivalent to $\frac{\sqrt{5}f_n + 2f_{n+1} - f_n}{2} > \sqrt{5}f_n$. This inequality together with (c) yields the desired result. By (a) with n even, $\phi > C_n$ is equivalent to $2\sqrt{5}f_n > \sqrt{5}f_n + 2f_{n+1} - f_n$, which is the same as $\frac{2}{\sqrt{5}f_n + 2f_{n+1} - f_n} > \frac{1}{\sqrt{5}f_n}$. This inequality and (c) yield the desired result. Finally, properties (d) and (e) give the desired result—which, with respect to the simple continued fraction algorithm, means that the golden mean is half silver! ∎

Observe that our main result, Corollary 6, rules out ϕ from having three consecutive gold convergents, but it does not exclude two out of every three consecutive convergents being gold for some ω. Can you find an irrational number with repeating assay sequence BGG or SGG?

The irrational number α is said to be *quadratic* if it is the root of a quadratic with integer coefficients. The golden mean and the radicals of Example 11 are such numbers. In 1770, Joseph Lagrange showed that ω's partial denominators are periodic if and only if ω is quadratic [8]. A natural conjecture is that the sequence of any quadratic irrational number's convergents is periodic with respect to being bronze, silver, and gold. Since using the partition benchmarks for bronze, silver, and gold as given by 1, $\frac{1}{2}$, and $\frac{1}{\sqrt{5}}$ seems somewhat arbitrary, one might intuitively doubt this conjecture. However, we leave the reader with a last question: before your CAS's precision in representing real numbers fizzles, can you find a quadratic irrational that fails to suggest convergent periodicity with respect to bronze, silver, and gold?

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