

A LOCAL MAZUR-ULAM THEOREM

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ABSTRACT. We prove a local version of the Mazur-Ulam theorem.

1. INTRODUCTION

In this paper we consider isometries between subsets of normed spaces. The Mazur-Ulam theorem asserts that an isometry from a normed space onto a normed space is real-linear up to translation (cf. [3, 4]). An isometry from a normed space *into* a normed space need not be real-linear up to translation (cf. [1]). An isometry from an open set U_1 of a normed space onto an open set U_2 need not be extended to a real-linear map up to translation (see Example 2.3). We show that if U_1 is star-shaped, then the isometry is extended to a real-linear map up to translation between the underlying normed spaces. We also consider maps defined on a subset of a normed space which is not necessarily open.

Throughout the paper B denotes a real normed space. For a subset X of B , $\text{Int}(X)$ is the interior of X . A star-shaped subset K with a center c of B is a set which satisfies that $tc + (1 - t)x \in K$ for every $x \in K$ and $0 \leq t \leq 1$. Let $a \in B$ and $\varepsilon > 0$. The open ball $\{x \in B : \|x - a\| < \varepsilon\}$ is denoted by $B_\varepsilon(a)$ and $\overline{B_\varepsilon(a)}$ its closure in B . For a pair a and b in B the set $\{x \in B : ta + (1 - t)b, 0 \leq t \leq 1\}$ is said to be a segment between a and b and is denoted by $[a, b]$.

2. ISOMETRIES BETWEEN OPEN SETS

We begin by showing a preliminary lemma. We prove it by making use of an idea of Väisälä [4]

Lemma 2.1. *Let $c \in B$ and a map $\psi : B \rightarrow B$ be defined as $\psi(z) = 2c - z$. Suppose that L is a non-empty bounded subset of B such that $c \in L$ and $\psi(L) = L$. If \mathcal{T} is a surjective isometry from L onto itself. Then $\mathcal{T}(c) = c$.*

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Proof. Let W be the set of all surjective isometries from L onto itself. Note that W is not empty since the identity function is in W . Let

$$\lambda = \sup\{\|g(c) - c\| : g \in W\}.$$

Since L is bounded $\lambda < \infty$. We will show that $\lambda = 0$. Suppose that $g \in W$. Let $g^* = g^{-1} \circ \psi \circ g$. Then $g^* \in W$. Hence

$$\begin{aligned} \lambda &\geq \|g^*(c) - c\| = \|g^{-1} \circ \psi \circ g(c) - c\| \\ &= \|\psi \circ g(c) - g(c)\| = 2\|g(c) - c\|. \end{aligned}$$

It follows that $\lambda \geq 2\lambda$ since g can be chosen arbitrary, hence $\lambda = 0$. \square

A real vector space with a metric $d(\cdot, \cdot)$ satisfying $d(a+u, b+u) = d(a, b)$ for all a, b, u , and for which addition and scalar multiplication are jointly continuous is called a metric real vector space.

Lemma 2.2. *Let \mathcal{B}_1 be a real normed space and \mathcal{B}_2 a metric real vector space with a metric $d(\cdot, \cdot)$. Suppose that U_1 and U_2 are non-empty open subsets of \mathcal{B}_1 and \mathcal{B}_2 respectively. Suppose that \mathcal{T} is a surjective isometry ($d(\mathcal{T}(a), \mathcal{T}(b)) = \|a - b\|$ for every $a, b \in U_1$) from U_1 onto U_2 and $f, g \in U_1$. If f and g satisfy the equation $(1-r)f + rg \in U_1$ for every r with $0 \leq r \leq 1$, then the equality*

$$\mathcal{T}\left(\frac{f+g}{2}\right) = \frac{\mathcal{T}(f) + \mathcal{T}(g)}{2}$$

holds.

Proof. Let $h, h' \in U_1$. Suppose that there exists $\varepsilon > 0$ which satisfies that $\frac{\|h-h'\|}{2} < \varepsilon$, and

$$\{a \in \mathcal{B}_1 : \|a - h\| < \varepsilon, \|a - h'\| < \varepsilon\} \subset U_1,$$

$$\{u \in \mathcal{B}_2 : d(u, \mathcal{T}(h)) < \varepsilon, d(u, \mathcal{T}(h')) < \varepsilon\} \subset U_2.$$

We will show that $\mathcal{T}\left(\frac{h+h'}{2}\right) = \frac{\mathcal{T}(h) + \mathcal{T}(h')}{2}$. Set $r = \frac{\|h-h'\|}{2}$ and let

$$L_1 = \{a \in \mathcal{B}_1 : \|a - h\| = r = \|a - h'\|\},$$

$$L_2 = \{u \in \mathcal{B}_2 : d(u, \mathcal{T}(h)) = r = d(u, \mathcal{T}(h'))\}.$$

Set also $c_1 = \frac{h+h'}{2}$ and $c_2 = \frac{\mathcal{T}(h) + \mathcal{T}(h')}{2}$. Then we have $\mathcal{T}(L_1) = L_2$, $c_1 \in L_1 \subset U_1$, and $L_2 \subset U_2$. Let

$$\psi_2(y) = 2c_2 - y \quad (y \in \mathcal{B}_2).$$

Then ψ_2 is an isometry and $\psi_2(L_2) = L_2$.

Let $Q = \mathcal{T}^{-1} \circ \psi_2 \circ \mathcal{T}$. Then Q is well-defined and is a surjective isometry from L_1 onto itself. Then by Lemma 2.1 $Q(c_1) = c_1$. Henceforth $\mathcal{T}(c_1) = c_2$.

We assume that f and g are as described. Let

$$K = \{(1-r)f + rg : 0 \leq r \leq 1\}.$$

Since K and $\mathcal{T}(K)$ are compact, there is $\varepsilon > 0$ with

$$\begin{aligned} \inf\{\|a - b\| : a \in K, b \in \mathcal{B}_1 \setminus U_1\} &> \varepsilon, \\ \inf\{d(u, v) : u \in \mathcal{T}(K), v \in \mathcal{B}_2 \setminus U_2\} &> \varepsilon. \end{aligned}$$

Then for every $h \in K$ we have

$$\{a \in \mathcal{B}_1 : \|a - h\| < \varepsilon\} \subset U_1$$

and

$$\{u \in \mathcal{B}_2 : \|u - \mathcal{T}(h)\| < \varepsilon\} \subset U_2.$$

Choose a natural number n with $\frac{\|f-g\|}{2^n} < \varepsilon$. Let

$$h_k = \frac{k}{2^n}(g - f) + f$$

for each $0 \leq k \leq 2^n$. By the first part of the proof we have

$$\mathcal{T}(h_k) + \mathcal{T}(h_{k+2}) - 2\mathcal{T}(h_{k+1}) = 0 \quad (k)$$

holds for $0 \leq k \leq 2^n - 2$. For $0 \leq k \leq 2^n - 4$, adding the equations (k) , 2 times of $(k+1)$, and $(k+2)$ we have

$$\mathcal{T}(h_k) + \mathcal{T}(h_{k+4}) - 2\mathcal{T}(h_{k+2}) = 0,$$

whence the equality

$$\mathcal{T}\left(\frac{f+g}{2}\right) = \frac{\mathcal{T}(f) + \mathcal{T}(g)}{2}$$

holds by induction on n . □

An isometry between open sets of normed spaces need not be extended to a linear map up to translation.

Example 2.3. Let $\mathcal{X} = \{x, y\}$ be a compact Hausdorff space consisting of two points and $C(\mathcal{X})$ denote the Banach algebra of all complex-valued continuous functions on \mathcal{X} . Let

$$\mathcal{U} = \{f \in C(\mathcal{X}) : \|f\| < 1\} \cup \{f \in C(\mathcal{X}) : \|f - f_0\| < 1\},$$

where $f_0 \in C(\mathcal{X})$ is defined as $f_0(x) = 0$, $f_0(y) = 10$. Suppose that

$$\mathcal{T} : \mathcal{U} \rightarrow \mathcal{U}$$

is defined as $\mathcal{T}(f) = f$ if $\|f\| < 1$ and $\mathcal{T}(f) = \tilde{f}$ if $\|f - f_0\| < 1$, where

$$\tilde{f}(t) = \begin{cases} -f(t), & t = x \\ f(t), & t = y. \end{cases}$$

Then \mathcal{T} is an isometry from \mathcal{U} onto itself, while it cannot be extended to a real linear isometry up to translation.

3. SUFFICIENT CONDITIONS FOR REAL-LINEARITY

Theorem 3.1. *Let B_1 and B_2 be real normed spaces. Let U_1 be a non-empty star-shaped open subset of B_1 . Suppose that $T : U_1 \rightarrow B_2$ is an isometry such that $T(U_1)$ is open in B_2 . Then there exists a surjective real-linear isometry \tilde{T}_0 from B_1 onto B_2 and $u \in B_2$ such that*

$$T(a) = \tilde{T}_0(a) + u \quad (a \in U)$$

holds.

Proof. Let a_0 be a center of U_1 ; i.e., $[a_0, x] \subset U_1$ for every $x \in U_1$. Let $\varphi : B_1 \rightarrow B_1$ be defined as $\varphi_1(x) = x + a_0$ ($x \in B_1$) and $\varphi_2 : B_2 \rightarrow B_2$ defined as $\varphi_2(x) = x - T(a_0)$ ($x \in B_2$). Then $\varphi_2 \circ T \circ \varphi_1 : U_1 - a_0 \rightarrow B_2$ is an isometry such that $\varphi_2 \circ T \circ \varphi_1(U_1 - a_0)$ is open in B_2 . We will show that $\varphi_2 \circ T \circ \varphi_1$ is extended to a surjective real-linear isometry \tilde{T}_0 from B_1 onto B_2 . It will follow that the conclusion holds. Let $V_1 = U_1 - a_0$ and $T_0 = \varphi_2 \circ T \circ \varphi_1$. There is $r > 0$ with $B_{3r}(0) \subset V_1$. Let $a \in V_1$. Since 0 is a center of V_1 , $ta = ta + (1-t)0 \in V_1$. Then by Lemma 2.2

$$T_0\left(\frac{a}{2}\right) = T_0\left(\frac{a+0}{2}\right) = \frac{T_0(a) + T_0(0)}{2} = \frac{T_0(a)}{2}$$

holds. For every $0 \leq s_1 \leq 1$ and $0 \leq s_2 \leq 1$, $t(s_1a) + (1-t)(s_2a) \in V_1$, hence

$$T_0\left(\frac{s_1a + s_2a}{2}\right) = \frac{T_0(s_1a) + T_0(s_2a)}{2}$$

holds. Applying the above equation and by induction on n

$$T_0\left(\frac{m}{2^n}a\right) = \frac{m}{2^n}T_0(a)$$

holds for every positive integer n and $m = 0, 1, \dots, 2^n$. It follows that

$$(3.1) \quad T_0(ta) = tT_0(a)$$

holds for every $a \in V_1$ and $0 \leq t \leq 1$. Let $a, b \in \overline{B_r(0)}$. Then $a + b \in B_{3r}(0) \subset V_1$ holds, and $ta + (1-t)b \in \overline{B_r(0)} \subset V_1$ ($0 \leq t \leq 1$) and (3.1) imply the equations

$$(3.2) \quad T_0(a+b) = 2T_0\left(\frac{a+b}{2}\right) = 2 \times \frac{T_0(a) + T_0(b)}{2} = T_0(a) + T_0(b)$$

by Lemma 2.2. Define a map $\tilde{T}_0 : B_1 \rightarrow B_2$ as

$$\tilde{T}_0(x) = \begin{cases} 0, & x = 0, \\ \frac{\|x\|}{r}T_0\left(\frac{r}{\|x\|}x\right), & x \neq 0. \end{cases}$$

We will show that $T_0(x) = \tilde{T}_0(x)$ for every $x \in V_1$. Let $x \in V_1$. If $x = 0$, then $T_0(x) = 0 = \tilde{T}_0(x)$. Suppose that $r \leq \|x\|$. Then by (3.1)

$$T_0\left(\frac{r}{\|x\|}x\right) = \frac{r}{\|x\|}T_0(x),$$

so $\tilde{T}_0(x) = T(x)$ holds. Suppose that $x \neq 0$ and $\|x\| < r$. Then

$$(3.3) \quad T_0(x) = T_0\left(\frac{\|x\|}{r}\frac{r}{\|x\|}x\right) = \frac{\|x\|}{r}T_0\left(\frac{r}{\|x\|}x\right) = \tilde{T}_0(x)$$

hold by (3.1).

We will show that

$$(3.4) \quad \tilde{T}_0(sx) = s\tilde{T}_0(x)$$

for every $x \in B_1$ and $s \in \mathbb{R}$. Let $a \in \overline{B_r(0)}$. Then $-a \in \overline{B_r(0)}$, and by (3.2) $T_0(-a) = -T_0(a)$ holds. Let $x \in B_1$ and $s \in \mathbb{R}$. If $x = 0$ or $s = 0$, then $\tilde{T}_0(sx) = s\tilde{T}_0(x)$ holds. Suppose that $x \neq 0$ and $s > 0$. Then

$$\tilde{T}_0(sx) = \frac{\|sx\|}{r}T_0\left(\frac{rs}{\|sx\|}x\right) = \frac{s\|x\|}{r}T_0\left(\frac{r}{\|x\|}x\right) = s\tilde{T}_0(x).$$

Suppose that $x \neq 0$ and $s < 0$. Then

$$\tilde{T}_0(sx) = \frac{-s\|x\|}{r}T_0\left(\frac{-r}{\|x\|}x\right) = \frac{s\|x\|}{r}T_0\left(\frac{r}{\|x\|}x\right) = s\tilde{T}_0(x)$$

since $\frac{r}{\|x\|}x \in \overline{B_r(0)}$.

We will show that

$$(3.5) \quad \tilde{T}_0(x+y) = \tilde{T}_0(x) + \tilde{T}_0(y)$$

holds for every $x, y \in B_1$. If $x+y=0$, then $y=-x$ and hence (3.5) holds by (3.4). Suppose that $x+y \neq 0$. If $x=0$ or $y=0$, then (3.5) holds since $\tilde{T}_0(0)=0$. Suppose that $x \neq 0$ and $y \neq 0$. Then by (3.2) and (3.3)

$$\begin{aligned} \tilde{T}_0\left(\frac{r}{\|x\|+\|y\|}x + \frac{r}{\|x\|+\|y\|}y\right) &= T_0\left(\frac{r}{\|x\|+\|y\|}x + \frac{r}{\|x\|+\|y\|}y\right) \\ &= T_0\left(\frac{r}{\|x\|+\|y\|}x\right) + T_0\left(\frac{r}{\|x\|+\|y\|}y\right) \\ &= \tilde{T}_0\left(\frac{r}{\|x\|+\|y\|}x\right) + \tilde{T}_0\left(\frac{r}{\|x\|+\|y\|}y\right) \end{aligned}$$

follows and (3.5) holds by (3.4).

We will show that \tilde{T}_0 is a surjection. Let $y \in B_2$. Then there is $r_0 > 0$ such that $\frac{y}{r_0} \in T_0(V_1)$ since $T_0(V_1)$ is open and $0 = T_0(0) \in T_0(V_1)$. Then there is $x_0 \in V_1$ with $T_0(x_0) = \frac{y}{r_0}$. It follows that $\tilde{T}_0(r_0x_0) = y$.

We will show that \tilde{T}_0 is an isometry. Let $x \in B_1$. If $x = 0$, then $\|\tilde{T}_0(x)\| = \|x\|$. Suppose that $x \neq 0$. Then

$$\tilde{T}_0(x) = \frac{\|x\|}{r} \tilde{T}_0\left(\frac{r}{\|x\|}x\right) = \frac{\|x\|}{r} T_0\left(\frac{r}{\|x\|}x\right)$$

hence

$$\|\tilde{T}_0(x)\| = \frac{\|x\|}{r} \|T_0\left(\frac{r}{\|x\|}x\right)\| = \frac{\|x\|}{r} \left\|\frac{r}{\|x\|}x - 0\right\| = \|x - 0\| = \|x\|$$

since T_0 is an isometry and $T_0 = \tilde{T}_0$ on V_1 . Thus \tilde{T}_0 is an isometry since \tilde{T}_0 is linear. \square

There are two preliminary lemmata for the following corollary. They may be standard, but proofs are included for the sake of completeness.

Lemma 3.2. *Suppose that U is a non-empty open subset of B and $p \in B \setminus U$. Then*

$$V_{p,U} = \cup_{x \in U} [p, x] \setminus \{p\}$$

is an open subset of B .

Proof. Suppose that $x_0 \in V_{p,U}$. Then there exist $y_0 \in U$ and $0 \leq t \leq 1$ with $x_0 = tp + (1-t)y_0$. Note that $t < 1$ holds, in fact, for $p \notin V_{p,U}$. Since U is open there exists $\varepsilon > 0$ with $B_\varepsilon(y_0) \subset U$. Then by a simple calculation

$$B_{(1-t)\varepsilon}(x_0) \subset V_{p,U}$$

holds. Hence $V_{p,U}$ is open for x_0 is arbitrary. \square

Lemma 3.3. *Suppose that X is a convex subset of B and $\text{Int}(X) \neq \emptyset$. Then $\text{Int}(X)$ is also convex and the closure $\overline{\text{Int}(X)}$ of $\text{Int}(X)$ contains X .*

Proof. We will show that $[a, b] \subset \text{Int}(X)$ for every pair a and b in $\text{Int}(X)$. Suppose that $a, b \in \text{Int}(X)$. Then there exists an open neighbourhood $U \subset \text{Int}(X)$ of b with $a \notin U$. Then Lemma 3.2 insures that $V_{a,U}$ is open. Since X is convex, $[a, x] \subset X$ for every $x \in U$, hence $V_{a,U} \subset X$, hence $V_{a,U} \subset \text{Int}(X)$ since $V_{a,U}$ is open. It follows that $[a, b] \subset \text{Int}(X)$ since $a \in \text{Int}(X)$; $\text{Int}(X)$ is convex.

Let $p \in X$. Let a be any element in $\text{Int}(X)$. Then there exists an open neighbourhood U of a such that $p \notin U \subset \text{Int}(X)$. Hence $V_{p,U} \subset \text{Int}(X)$ since $V_{p,U} \subset X$ for X is convex and $V_{p,U}$ is open by Lemma 3.2. Let $x_n = (1 - \frac{1}{n})p + \frac{1}{n}a$ for each positive integer n . Then $x_n \in V_{p,U}$ and $x_n \rightarrow p$ as $n \rightarrow \infty$. Then $p \in \overline{\text{Int}(X)}$. \square

Corollary 3.4. *Let B_1 and B_2 be real normed spaces. Let X be a convex subset of B_1 and $\text{Int}(X) \neq \emptyset$. Suppose that $T : X \rightarrow B_2$ is isometric and $T(\text{Int}(X))$ is open in B_2 . Then T is extended to a real-linear isometry up to translation.*

Proof. Since the restriction $T|_{\text{Int}(X)} : \text{Int}(X) \rightarrow B_2$ of T to $\text{Int}(X)$ satisfies the hypotheses of Theorem 3.1, $T|_{\text{Int}(X)}$ is extended to a surjective real-linear isometry up to translation $\widetilde{T|_{\text{Int}(X)}}$ from B_1 onto B_2 . Since T and $\widetilde{T|_{\text{Int}(X)}}$ are isometric and $X \subset \overline{\text{Int}(X)}$ holds by Lemma 3.3, $\widetilde{T|_{\text{Int}(X)}} = T$ on X . \square

An isometry from a star-shaped closed subset need not be extended to a real-linear map up to translation.

Example 3.5. Let $\mathbb{R}_{\max}^2 = \mathbb{R}^2$ as real linear spaces and the norm is defined as $\|(x, y)\| = \max\{|x|, |y|\}$ for $(x, y) \in \mathbb{R}_{\max}^2$. Let

$$X_1 = \{(x, y) \in \mathbb{R}_{\max}^2 : -1 \leq x \leq 0, |y| \leq -x\},$$

$$X_2 = \{(x, 0) \in \mathbb{R}_{\max}^2 : 0 \leq x \leq 1\}$$

and

$$X = X_1 \cup X_2.$$

Then X is a star-shaped closed subset of \mathbb{R}_{\max}^2 . Let

$$T : X \rightarrow \mathbb{R}_{\max}^2$$

be defined as

$$T((x, y)) = \begin{cases} (x, y), & (x, y) \in X_1, \\ (x, \sin x), & (x, y) \in X_2. \end{cases}$$

Then T is isometry and $T(\text{Int}(X))$ is open in \mathbb{R}_{\max}^2 . On the other hand T is not extended to a linear map since

$$T(-(1, 0)) = (-1, 0) \neq (-1, -\sin 1) = -T((1, 0))$$

and $T((0, 0)) = (0, 0)$.

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