he and Descartes carried on an intimate correspondence, each being a great admirer of the other. Little, however, could either have guessed that Christian, then only a child of three, would one day far outrank his father and would, in point of mathematical ability, rival his father's distinguished friend.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

To a question raised by the editor in connection with Professor Light's "Note on curves whose evolutes are similar curves" [*1920*, 303], as to the existence of other curves possessing the stated property, an answer is given below by Mr. Franklin, who derives in a simple way the result, stated by Puiseux, that an infinite set of such curves exists. A rather interesting incidental feature is found at the end of the paper, in the construction of a sort of geometrical "sum" of two curves as the locus of the sum (as determined by vectors starting at the origin) of points on the two curves at which tangents are parallel. The especial interest lies in the fact that exactly this process of composition of curves has been mentioned recently in an entirely different connection by Professor W. B. Carver[1]; the curves to which he applies the process are algebraic, while those of Mr. Franklin are transcendental.

Proofs of the law of tangents in plane trigonometry have been given recently by Cheney [*1920*, 53], Lovitt [*1920*, 465] and Epperson [*1921*, 71]. In a letter to the editor, from which an extract is printed, Professor Mathews calls attention to various other proofs of the law. It is clear, as Professor Mathews states, that other new proofs can be devised in considerable number from a suitable figure; it therefore seems desirable to bar consideration of further proofs, unless they involve new principles.

As the third discussion, we print a note on the nature of expository papers for presentation to the Association.

I. ON CURVES WHOSE EVOLUTES ARE SIMILAR CURVES.

BY PHILIP FRANKLIN, Princeton University.

In the July–September number of the MONTHLY [*1920*, 303] there appeared a discussion of curves whose evolutes are similar to themselves, by Professor Light. He found that the only curves having this property and having their intrinsic equations of the special type $AR^n + BS^m + C = 0$ were the logarithmic spiral and the cycloidal curves. The editor inquired whether any other curves possessed this property.

[1] "The failure of the Clifford chain," *American Journal of Mathematics*, vol. 42 (1920), p. 167.

That there is an infinite number of other classes of curves having the property in question was shown by Puiseux.[1] His method was similar to that used by Binet[2] to solve the special case of the problem where the evolute is equal to the original curve, although his work was independent of Binet's. The treatment here given follows closely that of Salmon.[3]

The simplicity of the derivation depends on the use of the relation between R (the radius of curvature) and t (the angle which the tangent at any point makes with a fixed tangent), instead of the ordinary intrinsic equation of the curve. Setting then

$$R = f(t) \tag{1}$$

as the equation of the original curve, we have

$$R_1 = af(t_1) \tag{2}$$

as the equation of its evolute, a being the ratio of similitude.

We may evidently set

$$t_1 = t + \alpha \tag{3}$$

and combining the last two equations with the relation:

$$\frac{dR}{dt} = R_1 \tag{4}$$

we obtain as the functional-differential equation for the determination of the function $f(t)$:

$$f'(t) = af(t + \alpha). \tag{5}$$

If the right member of this equation were capable of being expanded in a Taylor's series, the equation would have the *form* of a linear differential equation (of infinite order); hence we are led to seek a solution of the form $R = e^{mt}$ We find that this is a solution if the relation

$$m = ae^{m\alpha} \tag{6}$$

is satisfied. It is then evident that the equation

$$R = \Sigma C_i e^{m_i t}, \tag{7}$$

where the C_i are arbitrary, while the m_i are all roots of an equation of the form (6), defines a solution of (5). Whether (7) is the most general solution of (5) will not be discussed; it is probably the most general solution with no singularities except at infinity. Certain properties of the solutions of an equation similar to (5) have been obtained by Professor Fite.[4]

[1] M. Puiseux, Liouville's *Journal*, vol. 9, 1844, p. 377.

[2] J. Binet, Liouville's *Journal*, vol. 6, 1841, p. 61.

[3] G. Salmon, *Higher Plane Curves*, 1852, p. 280. The problem is not discussed in the later editions of the work.

[4] W. B. Fite, "Properties of the solutions of certain functional differential equations." *Bulletin of the American Mathematical Society*, vol. 26, pp. 245, 254.

As a particular case of (7) we have

$$R = Ae^{m_1 t} + Be^{m_2 t}, \tag{8}$$

where A and B are arbitrary constants, since a pair of values of a and α can be found such that m_1 and m_2 satisfy (6). If (8) is specialized by setting A and B equal, and taking a pair of conjugate complex numbers for m_1 and m_2, we obtain:

$$R = Ce^{mt} \cos nt, \tag{9}$$

which defines a logarithmic spiral when $n = 0$, and a cycloidal curve when $m = 0$, the cases obtained by Professor Light.

It is interesting to note that (6) has at most two real roots (since the equation obtained from it by differentiating both sides with respect to m evidently has at most one real root), but an infinity of complex roots.

Since the Cartesian equation of a curve given in the form (1) would be obtained by integrating the equations:

$$\begin{aligned} dx &= ds \cos t = R \cos t\, dt = f(t) \cos t\, dt, \\ dy &= ds \sin t = R \sin t\, dt = f(t) \sin t\, dt, \end{aligned} \tag{10}$$

and eliminating the parameter t, we see that the curve corresponding to (7) could be obtained from the curves corresponding to the separate terms of the sum by locating the points whose abscissas are the sums of the abscissas and whose ordinates are the sums of the ordinates of points on the component curves whose tangents are parallel, *i.e.*, points corresponding to the same value of t.

The problem admits of several generalizations. Puiseux[1] extended his solution to the case where it is merely required to find curves whose nth evolutes are similar to themselves. This extension presents no new difficulties. Binet[1] studied surfaces such that the locus of one of the two centers of curvature at each point was a surface equal to the original surface. We might also inquire whether there are any twisted curves such that one of their evolutes is similar; or such that the locus of centers of osculating spheres or of centers of curvature gives similar curves. This question is more difficult than that for the plane, owing to the greater number of constants determining a displacement, and the writer knows no successful method of attacking it, nor whether any particular solutions besides certain circular helices are known.

II. Geometric Proofs of the Law of Tangents.[2]

By R. M. Mathews, Wesleyan University.

Professor Lovitt's sixth proof, which uses the circumscribed circle, is given by: Killing und Hovestadt, *Handbuch des Mathematischen Unterrichts*, vol. 2, p. 27.

In *School Science and Mathematics*, vol. 15, pp. 798–801, I published an article, "Proofs of the Law of Tangents," in which I gave five different proofs.

[1] L. c.

[2] Extract from a letter to the editor.