Exemples

June 8, 2024

Définition. Soit $\mathcal{A} := (A, \theta)$ un anneau, on définit

$$\mathcal{A}^{op} := \left(A; \begin{cases} \{+; [opposit]; \star; 0; 1\} & \rightarrow & \bigcup \left\{\mathcal{P}\left(A^{n}\right) | n \in \mathbb{N}\right\} \\ x \in \{+; [opposit]; 0; 1\} & \mapsto & \theta(x) \\ \star & \mapsto & \left(\left\{\left(3; A; \left\{(0; a); (1; b); \left(2; \overline{\star ba}^{\mathcal{A}}\right)\right\}\right) | a, b \in A\right\}\right) \end{cases}\right)$$

Proposition 0.1. *Soit $\mathcal{A} := (A, \theta)$ un anneau, alors $\mathcal{A}^{op}$ est un anneau.*

Proof. Soit $\mathcal{V}_{\mathcal{A}}$ la fonction valeur de vérité associée à $\mathcal{A}$, soit $\mathcal{V}_{\mathcal{A}^{op}}$ la valeur de vérité associée à $\mathcal{A}^{op}$, soit

$$\psi := \begin{cases} \{+; [opposit]; \star; 0; 1\} & \rightarrow & \bigcup \left\{\mathcal{P}\left(A^{n}\right) | n \in \mathbb{N}\right\} \\ x \in \{+; [opposit]; 0; 1\} & \mapsto & \theta(x) \\ \star & \mapsto & \left\{\left(3; A; \left\{(0; a); (1; b); \left(2; \overline{\star ba}^{\mathcal{A}}\right)\right\}\right) | a, b \in A\right\} \end{cases}$$

on a

$$\begin{aligned} \mathcal{V}_{\mathcal{A}^{op}}(\forall a \forall b \forall c + +abc \sim +a + bc) &= \bigcap \left\{\mathcal{V}_{\mathcal{A}^{op}}(+ + abc \sim +a + bc) | a, b, c \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}^{op}}\left(\overline{+ + abc}^{\mathcal{A}^{op}} \sim \overline{+a + bc}^{\mathcal{A}^{op}}\right) | a, b, c \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}}\left(\overline{+ + abc}^{\mathcal{A}^{op}} \sim \overline{+a + bc}^{\mathcal{A}^{op}}\right) | a, b, c \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}}\left(\overline{+ + abc}^{\mathcal{A}} \sim \overline{+a + bc}^{\mathcal{A}}\right) | a, b, c \in A\right\} \\ &= \bigcap \{\mathcal{V}_{\mathcal{A}}(+ + abc \sim +a + bc) | a, b, c \in A\} \\ &= \mathcal{V}_{\mathcal{A}}(\forall a \forall b \forall c + +abc \sim +a + bc) \\ &= 1 \end{aligned}$$

on a

$$\begin{aligned} \mathcal{V}_{\mathcal{A}^{op}}(\forall a + a0 \sim a) &= \bigcap \left\{\mathcal{V}_{\mathcal{A}^{op}}(+a0 \sim a) | a \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}^{op}}\left(\overline{+a0}^{\mathcal{A}^{op}} \sim a\right) | a \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}}\left(\overline{+a0}^{\mathcal{A}^{op}} \sim a\right) | a \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}}\left(\overline{+a0}^{\mathcal{A}} \sim a\right) | a \in A\right\} \\ &= \bigcap \{\mathcal{V}_{\mathcal{A}}(+a0 \sim a) | a \in A\} \\ &= \mathcal{V}_{\mathcal{A}}(\forall a + a0 \sim a) \\ &= 1 \end{aligned}$$

on a

$$\begin{aligned} \mathcal{V}_{\mathcal{A}^{op}}(\forall a + 0a \sim a) &= \bigcap \left\{\mathcal{V}_{\mathcal{A}^{op}}(+0a \sim a) | a \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}^{op}}\left(\overline{+0a}^{\mathcal{A}^{op}} \sim a\right) | a \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}}\left(\overline{+0a}^{\mathcal{A}^{op}} \sim a\right) | a \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}}\left(\overline{+0a}^{\mathcal{A}} \sim a\right) | a \in A\right\} \\ &= \bigcap \{\mathcal{V}_{\mathcal{A}}(+0a \sim a) | a \in A\} \\ &= \mathcal{V}_{\mathcal{A}}(\forall a + 0a \sim a) \\ &= 1 \end{aligned}$$

on a

$$\begin{aligned} \mathcal{V}_{\mathcal{A}^{op}}(\forall a \forall b + ab \sim +ba) &= \bigcap \left\{\mathcal{V}_{\mathcal{A}^{op}}(+ab \sim +ba) | a, b \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}^{op}}\left(\overline{+ab}^{\mathcal{A}^{op}} \sim \overline{+ba}^{\mathcal{A}^{op}}\right) | a, b \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}}\left(\overline{+ab}^{\mathcal{A}^{op}} \sim \overline{+ba}^{\mathcal{A}^{op}}\right) | a, b \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}}\left(\overline{+ab}^{\mathcal{A}} \sim \overline{+ba}^{\mathcal{A}}\right) | a, b \in A\right\} \\ &= \bigcap \{\mathcal{V}_{\mathcal{A}}(+ab \sim +ba) | a, b \in A\} \\ &= \mathcal{V}_{\mathcal{A}}(\forall a \forall b + ab \sim +ba) \\ &= 1 \end{aligned}$$

on a

$$\begin{aligned} \mathcal{V}_{\mathcal{A}^{op}}(\forall a + a\,[opposit]\,a \sim 0) &= \bigcap \left\{\mathcal{V}_{\mathcal{A}^{op}}(+a\,[opposit]\,a \sim 0) | a \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}^{op}}\left(\overline{+a\,[opposit]\,a}^{\mathcal{A}^{op}} \sim \overline{0}^{\mathcal{A}^{op}}\right) | a \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}}\left(\overline{+a\,[opposit]\,a}^{\mathcal{A}^{op}} \sim \overline{0}^{\mathcal{A}^{op}}\right) | a \in A\right\} \\ &= \bigcap \left\{\mathcal{V}_{\mathcal{A}}\left(\overline{+a\,[opposit]\,a}^{\mathcal{A}} \sim \overline{0}^{\mathcal{A}}\right) | a \in A\right\} \\ &= \bigcap \{\mathcal{V}_{\mathcal{A}}(+a\,[opposit]\,a \sim 0) | a \in A\} \\ &= \mathcal{V}_{\mathcal{A}}(\forall a + a\,[opposit]\,a \sim 0) \\ &= 1 \end{aligned}$$

on a

$$
\begin{aligned}
\mathcal{V}_{\mathcal{A}^{op}}\left(\forall a\forall b\forall c\star\star abc\sim\star a\star bc\right) &= \bigcap\left\{\mathcal{V}_{\mathcal{A}^{op}}\left(\star\star abc\sim\star a\star bc\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}^{op}}\left(\overline{\star\star abc}^{\mathcal{A}^{op}}\sim\overline{\star a\star bc}^{\mathcal{A}^{op}}\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star\star abc}^{\mathcal{A}^{op}}\sim\overline{\star a\star bc}^{\mathcal{A}^{op}}\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star\overline{\star ab}^{\mathcal{A}^{op}}c}^{\mathcal{A}^{op}}\sim\overline{\star a\overline{\star bc}^{\mathcal{A}^{op}}}^{\mathcal{A}^{op}}\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&\overline{\star ab}^{\mathcal{A}^{op}}\\1&\mapsto&c\end{array}\right.\right)\sim\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&a\\1&\mapsto&\overline{\star bc}^{\mathcal{A}^{op}}\end{array}\right.\right)\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&a\\1&\mapsto&b\end{array}\right.\right)\\1&\mapsto&c\end{array}\right.\right)\sim\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&a\\1&\mapsto&\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&b\\1&\mapsto&c\end{array}\right.\right)\end{array}\right.\right)\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\theta\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&c\\1&\mapsto&\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&a\\1&\mapsto&b\end{array}\right.\right)\end{array}\right.\right)\sim\left(\theta\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&b\\1&\mapsto&c\end{array}\right.\right)\\1&\mapsto&a\end{array}\right.\right)\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\theta\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&c\\1&\mapsto&\left(\theta\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&b\\1&\mapsto&a\end{array}\right.\right)\end{array}\right.\right)\sim\left(\theta\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&\left(\theta\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&c\\1&\mapsto&b\end{array}\right.\right)\\1&\mapsto&a\end{array}\right.\right)\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\theta\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&c\\1&\mapsto&\overline{\star ba}^{\mathcal{A}}\end{array}\right.\right)\sim\left(\theta\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&\overline{\star cb}^{\mathcal{A}}\\1&\mapsto&a\end{array}\right.\right)\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star c\overline{\star ba}^{\mathcal{A}}}^{\mathcal{A}}\sim\overline{\star\overline{\star cb}^{\mathcal{A}}a}^{\mathcal{A}}\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star c\star ba}^{\mathcal{A}}\sim\overline{\star\star cba}^{\mathcal{A}}\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\star c\star ba\sim\star\star cba\right)|a,b,c\in A\right\}\\
&= \mathcal{V}_{\mathcal{A}}\left(\forall a\forall b\forall c\star c\star ba\sim\star\star cba\right)\\
&= 1
\end{aligned}
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on a

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\begin{aligned}
\mathcal{V}_{\mathcal{A}^{op}}\left(\forall a\star a1\sim a\right) &= \bigcap\left\{\mathcal{V}_{\mathcal{A}^{op}}\left(\star a1\sim a\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}^{op}}\left(\overline{\star a1}^{\mathcal{A}^{op}}\sim a\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star a1}^{\mathcal{A}^{op}}\sim a\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star a\overline{1}^{\mathcal{A}^{op}}}^{\mathcal{A}^{op}}\sim a\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&a\\1&\mapsto&\overline{1}^{\mathcal{A}^{op}}\end{array}\right.\right)\sim a\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&a\\1&\mapsto&\left(\psi\left(1\right)\right)\left(\emptyset;A;\emptyset\right)\end{array}\right.\right)\sim a\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&a\\1&\mapsto&\left(\theta\left(1\right)\right)\left(\emptyset;A;\emptyset\right)\end{array}\right.\right)\sim a\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&a\\1&\mapsto&\overline{1}^{\mathcal{A}}\end{array}\right.\right)\sim a\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\theta\left(\star\right)\right)\left(\left\{\begin{array}{ccc}2&\rightarrow&A\\0&\mapsto&\overline{1}^{\mathcal{A}}\\1&\mapsto&a\end{array}\right.\right)\sim a\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star\overline{1}^{\mathcal{A}}a}^{\mathcal{A}}\sim a\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star 1a}^{\mathcal{A}}\sim a\right)|a,b,c\in A\right\}\\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\star 1a\sim a\right)|a,b,c\in A\right\}\\
&= \mathcal{V}_{\mathcal{A}}\left(\forall a\star 1a\sim a\right)\\
&= 1
\end{aligned}
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on a

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\begin{aligned}
\mathcal{V}_{\mathcal{A}^{op}}\left(\forall a \star 1a \sim a\right) &= \bigcap\left\{\mathcal{V}_{\mathcal{A}^{op}}\left(\star 1a \sim a\right)|a,b,c \in A\right\} \\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}^{op}}\left(\overline{\star 1a}^{\mathcal{A}^{op}} \sim a\right)|a,b,c \in A\right\} \\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star 1a}^{\mathcal{A}^{op}} \sim a\right)|a,b,c \in A\right\} \\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star \overline{1}^{\mathcal{A}^{op}} a}^{\mathcal{A}^{op}} \sim a\right)|a,b,c \in A\right\} \\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & \overline{1}^{\mathcal{A}^{op}} \\ 1 & \mapsto & a \end{array}\right.\right) \sim a\right)|a,b,c \in A\right\} \\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & \left(\psi\left(1\right)\right)\left(\emptyset; A; \emptyset\right) \\ 1 & \mapsto & a \end{array}\right.\right) \sim a\right)|a,b,c \in A\right\} \\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & \left(\theta\left(1\right)\right)\left(\emptyset; A; \emptyset\right) \\ 1 & \mapsto & a \end{array}\right.\right) \sim a\right)|a,b,c \in A\right\} \\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & \overline{1}^{\mathcal{A}} \\ 1 & \mapsto & a \end{array}\right.\right) \sim a\right)|a,b,c \in A\right\} \\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\theta\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & a \\ 1 & \mapsto & \overline{1}^{\mathcal{A}} \end{array}\right.\right) \sim a\right)|a,b,c \in A\right\} \\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star a \overline{1}^{\mathcal{A}}}^{\mathcal{A}} \sim a\right)|a,b,c \in A\right\} \\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star a 1}^{\mathcal{A}} \sim a\right)|a,b,c \in A\right\} \\
&= \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\star a 1 \sim a\right)|a,b,c \in A\right\} \\
&= \mathcal{V}_{\mathcal{A}}\left(\forall a \star a 1 \sim a\right) \\
&= 1
\end{aligned}
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on a

$$
\begin{aligned}
& \mathcal{V}_{\mathcal{A}^{op}}\left(\forall a \forall b \forall c \star + abc \sim + \star ac \star bc\right) \\
= & \bigcap\left\{\mathcal{V}_{\mathcal{A}^{op}}\left(\star + abc \sim + \star ac \star bc\right)|a,b,c \in A\right\} \\
= & \bigcap\left\{\mathcal{V}_{\mathcal{A}^{op}}\left(\overline{\star + abc}^{\mathcal{A}^{op}} \sim \overline{+ \star ac \star bc}^{\mathcal{A}^{op}}\right)|a,b,c \in A\right\} \\
= & \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star + abc}^{\mathcal{A}^{op}} \sim \overline{+ \star ac \star bc}^{\mathcal{A}^{op}}\right)|a,b,c \in A\right\} \\
= & \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star \overline{+ab}^{\mathcal{A}^{op}} c}^{\mathcal{A}^{op}} \sim \overline{+ \overline{\star ac}^{\mathcal{A}^{op}} \overline{\star bc}^{\mathcal{A}^{op}}}^{\mathcal{A}^{op}}\right)|a,b,c \in A\right\} \\
= & \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & \overline{+ab}^{\mathcal{A}^{op}} \\ 1 & \mapsto & c \end{array}\right.\right) \sim \left(\psi\left(+\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & \overline{\star ac}^{\mathcal{A}^{op}} \\ 1 & \mapsto & \overline{\star bc}^{\mathcal{A}^{op}} \end{array}\right.\right)\right)|a,b,c \in A\right\} \\
= & \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & \left(\psi\left(+\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & a \\ 1 & \mapsto & b \end{array}\right.\right) \\ 1 & \mapsto & c \end{array}\right.\right) \sim \left(\psi\left(+\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & \left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & a \\ 1 & \mapsto & c \end{array}\right.\right) \\ 1 & \mapsto & \left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & b \\ 1 & \mapsto & c \end{array}\right.\right) \end{array}\right.\right)\right)|a,b,c \in A\right\} \\
= & \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & \left(\theta\left(+\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & a \\ 1 & \mapsto & b \end{array}\right.\right) \\ 1 & \mapsto & c \end{array}\right.\right) \sim \left(\theta\left(+\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & \left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & a \\ 1 & \mapsto & c \end{array}\right.\right) \\ 1 & \mapsto & \left(\psi\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & b \\ 1 & \mapsto & c \end{array}\right.\right) \end{array}\right.\right)\right)|a,b,c \in A\right\} \\
= & \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\theta\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & c \\ 1 & \mapsto & \left(\theta\left(+\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & a \\ 1 & \mapsto & b \end{array}\right.\right) \end{array}\right.\right) \sim \left(\theta\left(+\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & \left(\theta\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & c \\ 1 & \mapsto & a \end{array}\right.\right) \\ 1 & \mapsto & \left(\theta\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & c \\ 1 & \mapsto & b \end{array}\right.\right) \end{array}\right.\right)\right)|a,b,c \in A\right\} \\
= & \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\left(\theta\left(\star\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & c \\ 1 & \mapsto & \overline{+ab}^{\mathcal{A}} \end{array}\right.\right) \sim \left(\theta\left(+\right)\right)\left(\left\{\begin{array}{ccc} 2 & \rightarrow & A \\ 0 & \mapsto & \overline{\star ca}^{\mathcal{A}} \\ 1 & \mapsto & \overline{\star cb}^{\mathcal{A}} \end{array}\right.\right)\right)|a,b,c \in A\right\} \\
= & \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\overline{\star c \overline{+ab}^{\mathcal{A}}}^{\mathcal{A}} \sim \overline{+ \overline{\star ca}^{\mathcal{A}} \overline{\star cb}^{\mathcal{A}}}^{\mathcal{A}}\right)|a,b,c \in A\right\} \\
= & \bigcap\left\{\mathcal{V}_{\mathcal{A}}\left(\star c + ab \sim + \star ca \star cb\right)|a,b,c \in A\right\} \\
= & \mathcal{V}_{\mathcal{A}}\left(\forall c \forall a \forall b \star c + ab \sim + \star ca \star cb\right) \\
= & 1
\end{aligned}
$$

on a

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\begin{aligned}
& \mathcal{V}_{\mathcal{A}^{op}}\left(\forall a \forall b \forall c \star a + bc \sim + \star ab \star ac\right) \\
= \; & \bigcap \left\{ \mathcal{V}_{\mathcal{A}^{op}}\left(\star a + bc \sim + \star ab \star ac\right) | a,b,c \in A \right\} \\
= \; & \bigcap \left\{ \mathcal{V}_{\mathcal{A}^{op}}\left(\overline{\star a + bc}^{\mathcal{A}^{op}} \sim \overline{+ \star ab \star ac}^{\mathcal{A}^{op}}\right) | a,b,c \in A \right\} \\
= \; & \bigcap \left\{ \mathcal{V}_{\mathcal{A}}\left(\overline{\star a + bc}^{\mathcal{A}^{op}} \sim \overline{+ \star ab \star ac}^{\mathcal{A}^{op}}\right) | a,b,c \in A \right\} \\
= \; & \bigcap \left\{ \mathcal{V}_{\mathcal{A}}\left(\overline{\star a \overline{+bc}^{\mathcal{A}^{op}}}^{\mathcal{A}^{op}} \sim \overline{+ \overline{\star ab}^{\mathcal{A}^{op}} \overline{\star ac}^{\mathcal{A}^{op}}}^{\mathcal{A}^{op}}\right) | a,b,c \in A \right\} \\
= \; & \bigcap \left\{ \mathcal{V}_{\mathcal{A}}\left((\psi(\star)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & a \\ 1 & \mapsto & \overline{+bc}^{\mathcal{A}^{op}} \end{array} \right. \right) \sim (\psi(+)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & \overline{\star ab}^{\mathcal{A}^{op}} \\ 1 & \mapsto & \overline{\star ac}^{\mathcal{A}^{op}} \end{array} \right. \right) \right) | a,b,c \in A \right\} \\
= \; & \bigcap \left\{ \mathcal{V}_{\mathcal{A}}\left((\psi(\star)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & a \\ 1 & \mapsto & (\psi(+)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & b \\ 1 & \mapsto & c \end{array} \right. \right) \end{array} \right. \right) \sim (\psi(+)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & (\psi(\star)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & a \\ 1 & \mapsto & b \end{array} \right. \right) \\ 1 & \mapsto & (\psi(\star)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & a \\ 1 & \mapsto & c \end{array} \right. \right) \end{array} \right. \right) \right) | a,b,c \in A \right\} \\
= \; & \bigcap \left\{ \mathcal{V}_{\mathcal{A}}\left((\psi(\star)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & a \\ 1 & \mapsto & (\theta(+)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & b \\ 1 & \mapsto & c \end{array} \right. \right) \end{array} \right. \right) \sim (\psi(+)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & (\theta(\star)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & b \\ 1 & \mapsto & a \end{array} \right. \right) \\ 1 & \mapsto & (\theta(\star)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & c \\ 1 & \mapsto & a \end{array} \right. \right) \end{array} \right. \right) \right) | a,b,c \in A \right\} \\
= \; & \bigcap \left\{ \mathcal{V}_{\mathcal{A}}\left((\psi(\star)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & a \\ 1 & \mapsto & \overline{+bc}^{\mathcal{A}} \end{array} \right. \right) \sim (\psi(+)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & \overline{\star ba}^{\mathcal{A}} \\ 1 & \mapsto & \overline{\star ca}^{\mathcal{A}} \end{array} \right. \right) \right) | a,b,c \in A \right\} \\
= \; & \bigcap \left\{ \mathcal{V}_{\mathcal{A}}\left((\theta(\star)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & \overline{+bc}^{\mathcal{A}} \\ 1 & \mapsto & a \end{array} \right. \right) \sim (\theta(+)) \left(\left\{ \begin{array}{ccc} 2 & \to & A \\ 0 & \mapsto & \overline{\star ba}^{\mathcal{A}} \\ 1 & \mapsto & \overline{\star ca}^{\mathcal{A}} \end{array} \right. \right) \right) | a,b,c \in A \right\} \\
= \; & \bigcap \left\{ \mathcal{V}_{\mathcal{A}}\left(\overline{\star \overline{+bc}^{\mathcal{A}} a}^{\mathcal{A}} \sim \overline{+ \overline{\star ba}^{\mathcal{A}} \overline{\star ca}^{\mathcal{A}}}^{\mathcal{A}}\right) | a,b,c \in A \right\} \\
= \; & \bigcap \left\{ \mathcal{V}_{\mathcal{A}}\left(\star + bca \sim + \star ba \star ca\right) | a,b,c \in A \right\} \\
= \; & \mathcal{V}_{\mathcal{A}}\left(\forall b \forall c \forall a \star + bca \sim + \star ba \star ca\right) \\
= \; & 1
\end{aligned}
$$

CQFD. □