2166. On the Formula of the Mean
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null diagonal. Suppose that the first r columns of the transforming matrix are known. These correspond to r mutually perpendicular generators of a cone in n-dimensional space. For the $(r+1)$-th, we may take any of the ∞^{n-r-2} generators of the cone which is the intersection of the given cone with the sub-space orthogonal to the first r generators. The total number of degrees of freedom is therefore

$$\sum_{r=0}^{n-2} (n-r-2) = \tfrac{1}{2}(n-1)(n-2).$$

Since a Hermitian matrix can be transformed by a unitary matrix into a real diagonal form, it is clear that it can also be transformed by a unitary matrix into a real symmetric matrix with null diagonal.

The sum of the squared moduli of the elements of a matrix A is invariant under unitary transformations, since it is the trace of AA'. Hence, in particular, if $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of a Hermitian matrix A, and

$$\lambda_1 + \lambda_2 + \lambda_3 = 0,$$

then if A is transformed into a real matrix with null diagonal by a unitary matrix, the elements of the transform are bounded by the numbers

$$\pm\{\tfrac{1}{2}(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2)\}^{\frac{1}{2}}.$$

J. D. Weston.

2166. *On the formula of the mean.*

It is not an infrequent rider on the formula of the mean

$$f(a+h) - f(a) = hf'(a+\theta h) \quad \text{.............................(1)}$$

to ask for a proof that, when θ is independent of both a, h, i.e. is an absolute constant, then the function f must be quadratic or possibly linear. The usual proof relies on partial differentiation in a, h and presupposes f differentiable to the second or even the third order. It is, however, possible to obtain the result without further differentiation, and, in fact, we may replace (1) by the less informative identity

$$f(a+h) - f(a) = h\phi(a+\theta h), \quad \text{............................(2)}$$

where all that I suppose is that f, ϕ are defined for all real values of the variable and are one-valued : this simplifies the essential argument.

Interchanging a, $a+h$ in (2) gives

$$f(a) - f(a+h) = -h\phi\{a+(1-\theta)h\},$$

so that

$$\phi(a+\theta h) = \phi\{a+(1-\theta)h\}.$$

If $\theta \neq \frac{1}{2}$, we can solve (in a, h)

$$a+\theta h = x, \quad a+(1-\theta)h = y$$

for arbitrary x, y. This gives $\phi(x) = \phi(y)$ for any x, y, and so $\phi(x)$ is a constant (A, say). Then, from (2),

$$f(a+h) - A(a+h) = f(a) - Aa,$$

so that $f(x) - Ax$ is again a constant (B, say). Thus, when $\theta \neq \frac{1}{2}$, $f(x)$ has the linear form $Ax+B$ (and θ is irrelevant).

When $\theta = \frac{1}{2}$, the defining formula is

$$f(a+h) - f(a) = h\phi(a+\tfrac{1}{2}h). \quad \text{.............................(3)}$$

Giving (a, h) the pairs of values $(a, -h)$, $(a-h, 2h)$ we also have

$$f(a-h)-f(a)=-h\phi(a-\tfrac{1}{2}h), \quad f(a+h)-f(a-h)=2h\phi(a).$$

Eliminating f between these three identities gives

$$\phi(a+\tfrac{1}{2}h)+\phi(a-\tfrac{1}{2}h)=2\phi(a), \qquad \text{.......(4)}$$

and therefore

$$\phi(a+\tfrac{1}{2}h)+\phi(a-\tfrac{1}{2}h)=\phi(a+\tfrac{1}{2}k)+\phi(a-\tfrac{1}{2}k), \qquad \text{.......(5)}$$

since h is absent from the right of (4).

With $a=\frac{1}{2}(x+t)$, $h=x+t$, $k=x-t$ we can rewrite (5) as

$$\phi(x+t)-\phi(x)=\phi(t)-\phi(0)\equiv\psi(t), \text{ say.} \qquad \text{.......(6)}$$

Then, from (3),

$$f(x+y)-f(x)-f(y)+f(0)=y\{\phi(x+\tfrac{1}{2}y)-\phi(\tfrac{1}{2}y)\}=y\psi(x).$$

By symmetry in x, y this must also equal $x\psi(y)$, and so

$$\psi(x)/x=\psi(y)/y=\text{a constant } (A, \text{ say}).$$

Thus, from (6),

$$\phi(x+t)-\phi(x)=At,$$

which, as in the first part, gives

$$\phi(x)=Ax+B,$$

with B a second constant. Then, from (3) again,

$$f(x)-f(y)=(x-y)\phi(\tfrac{1}{2}x+\tfrac{1}{2}y)$$
$$=(x-y)\{\tfrac{1}{2}A(x+y)+B\},$$

i.e. $\qquad f(x)-\frac{1}{2}Ax^2-Bx=f(y)-\frac{1}{2}Ay^2-By=\text{a constant } (C, \text{ say}).$

This gives $f(x)$ the quadratic form $\frac{1}{2}Ax^2+Bx+C$, and completes the proof.

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2167. *Pan-Magic squares of even order.*

N. and W. J. Chater have proved (*Mathematical Gazette*, XXXIII, No. 304) that the determinant of a pan-magic square of even order is zero. An alternative proof, which yields further information concerning the properties of such squares, is set out below.

A pan-magic square of order $2n$ is of the form

$$\begin{array}{ccc|ccc|c}
a_1 a_2 & \cdots\cdots & a_n & A_1' A_2' & \cdots\cdots & A_n' & \lambda_1 \\
b_1 b_2 & \cdots\cdots & b_n & B_1' B_2' & \cdots\cdots & B_n' & \lambda_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
k_1 k_2 & \cdots\cdots & k_n & K_1' K_2' & \cdots\cdots & K_n' & \lambda_n \\
\hline
A_1 A_2 & \cdots\cdots & A_n & a_1' a_2' & \cdots\cdots & a_n' & -\lambda_1 \\
B_1 B_2 & \cdots\cdots & B_n & b_1' b_2' & \cdots\cdots & b_n' & -\lambda_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
K_1 K_2 & \cdots\cdots & K_n & k_1' k_2' & \cdots\cdots & k_n' & -\lambda_n
\end{array} \qquad \text{.......(i)}$$

dashes denoting complements with respect to S/n, where S is the square constant. Thus $A_1'=(S/n)-A_1$.

We have

$$\sum_{i=1}^{n} A_i=S-\sum_{i=1}^{n} a_i'$$
$$=S-\sum_{i=1}^{n}\left(\frac{S}{n}-a_i\right),$$