for some point M in plane. Choosing $M \equiv O$, we are left with proving that

$$\overrightarrow{OD} \cdot \overrightarrow{AY} + \overrightarrow{OF} \cdot \overrightarrow{XA} = 0. \tag{5}$$

Let $\overrightarrow{OY} = \overrightarrow{OA} + \lambda(\overrightarrow{OA} - \overrightarrow{OB}) = \overrightarrow{OE} + \mu(\overrightarrow{OE} - \overrightarrow{OF})$. As $\vec{a}$ and $\vec{b}$ are linearly independent, it follows that $\lambda = \beta(1-\alpha)/(\alpha-\beta)$, i.e. $\overrightarrow{AY} = \overrightarrow{OY} - \overrightarrow{OA} = \lambda(\overrightarrow{OA} - \overrightarrow{OB}) = \beta(1-\alpha)/(\alpha-\beta)(\vec{a}-\vec{b})$.

In a similar fashion we get $\overrightarrow{XA} = \gamma(1-\alpha)/(\alpha-\gamma)(\vec{c}-\vec{a})$. Consequently, (5) becomes

$$\begin{aligned}
&\gamma\vec{c}\,\frac{\beta(1-\alpha)}{\alpha-\beta}(\vec{a}-\vec{b}) + \beta\vec{b}\,\frac{\gamma(1-\alpha)}{\alpha-\gamma}(\vec{c}-\vec{a}) = 0\\
&\Leftrightarrow \frac{\varphi-\psi}{\alpha-\beta} + \frac{\psi-\theta}{\alpha-\gamma} = 0\\
&\Leftrightarrow (\alpha-\gamma)(\varphi-\psi) + (\alpha-\beta)(\psi-\theta) = 0\\
&\Leftrightarrow \alpha(\varphi-\theta) + \beta(\theta-\psi) + \gamma(\psi-\varphi) = 0,
\end{aligned}$$

i.e. (5) is equivalent to (4), and the conclusion follows in this case.

Finally, in the case in which we do not have a proper perspective axis (i.e. this is thrown to infinity), we infer that $\alpha = \beta = \gamma$. Hence $\triangle ABC$ and $\triangle EFD$ are homothetic with constant α. Now, the orthology centers of $\triangle EFD$ and $\triangle ABC$ become the orthocenter of $\triangle EFD$ and the orthocenter of $\triangle ABC$, respectively.

Since these triangles are homothetic, it follows that O, S and P lie on the same line. Moreover, $\overrightarrow{OS} = \alpha\,\overrightarrow{OP}$. This completes the proof of the Theorem.

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REFERENCE

[CG] Coxeter, H., S., M. and Greitzer, S., L., *Geometry Revisited*, Random House 19 (1967).

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The Existence of a Triangle with Prescribed Angle Bisector Lengths

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Given three arbitrary positive numbers m, n, p, does there exist a triangle with angle bisectors of length m, n, p? The answer is YES! Moreover, the triangle is unique up to an isometry. This contrasts with related results for the triangle. The median lengths m, n, p satisfy $m < n + p$ (and the two other cyclic inequalities) and the altitude lengths satisfy $1/m < 1/n + 1/p$; thus the lengths m, n, p cannot be arbitrary in these cases.

The referee has kindly submitted to us several bibliographic notes which show the long history of this problem. Brocard proposed in 1875 in the *Nouvelle Correspondance Mathématique* the problem of constructing such a triangle. The problem was reduced to that of solving an equation of degree 16 by F. J. Van Den Berg (Nieuw Archief voor Wiskunde, 1889). In 1896, P. Barbarin showed in *Mathesis* that the equation can be chosen to be of degree 14 and that it is irreducible in general. He also showed that this equation becomes an irreducible cubic when two of the angle bisector lengths are equal—which shows the impossibility of a Euclidean construction for the triangle. More detailed references can be found in **[1]**, **[2]**.

We now prove the announced result. First, we derive some needed formulas. If a, b, c are the side lengths, m, n, p the angle bisector lengths and s the semiperimeter of a triangle, then we have

$$m = \frac{2}{b+c}\sqrt{bcs(s-a)} \tag{1}$$

with similar formulas for n and p. We prove (1) by an area argument (see FIGURE

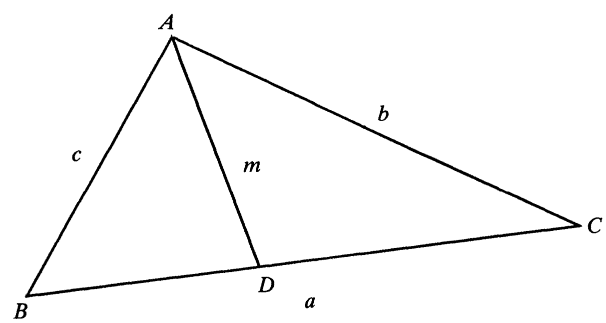

Figure 1

1). If $S(MNP)$ denotes the arc of triangle MNP, then

$$2\,S(ABC) = 2\,S(ABD) + 2\,S(ACD)$$

so that

$$bc\sin(A) = bm\sin(A/2) + cm\sin(A/2)$$

and hence

$$m = \frac{bc}{b+c}\frac{\sin(A)}{\sin(A/2)} = \frac{2bc}{b+c}\cos(A/2) = \frac{2bc}{b+c}\sqrt{s(s-a)/bc}\,.$$

One can easily check that

$$\left[b + c \pm \frac{a(b-c)}{b+c}\right]^2 = 4m^2 + \left[a \pm (b-c)\right]^2 \tag{2}$$

and (2) gives

$$b + c = \sqrt{m^2 + (s-b)^2} + \sqrt{m^2 + (s-c)^2}\,. \tag{3}$$

We get two related equalities for $c + a$ and $a + b$. With the substitutions

$$x = s - a, \qquad y = s - b, \qquad z = s - c \tag{4}$$

the relation (3) can be rewritten as

$$x = \left[\sqrt{m^2 + y^2} - y\right]/2 + \left[\sqrt{m^2 + z^2} - z\right]/2 = f(y, m) + f(z, m) \tag{5}$$

where $f(u, v) = [\sqrt{u^2 + v^2} - u]/2$.

Given arbitrary positive real numbers m, n, p, let $F\colon [0, \infty)^3 \to (0, \infty)^3$ be defined by

$$F(x, y, z) = (f(y, m) + f(z, m), f(z, n) + f(x, n), f(x, p) + f(y, p)).$$

Taking (5) into account, we get

$$F(x, y, z) = (x, y, z) \tag{6}$$

whenever x, y, z are as in (4) in a triangle with angle bisector lengths m, n, p. Conversely, if (6) holds, then in the triangle with sides lengths $y + z$, $z + x$, $x + y$ the equality (3) holds (together with the two analogous ones) so that m must be given by (1), in virtue of the monotonicity of the right side of (3) in the variable m. Hence the given problem is equivalent to the existence and uniqueness of a fixed point for F.

EXISTENCE. Since $f(u, v) \in [0, v/2]$ for nonnegative u, v, it follows that $F(K) \subseteq K$ where $K = [0, m] \times [0, n] \times [0, p]$. Note that $(0, 0, 0)$ is *not* a fixed point of F. Since K is a convex compact set in $\mathbb{R}^2$ and F is continuous, the existence follows by the Brouwer Fixed Point Theorem (see, for example, [**3**]).

UNIQUENESS. For $v \neq 0$, $u \neq t$, and $D = \sqrt{u^2 + v^2} + \sqrt{t^2 + v^2}$, we have

$$2|f(u, v) - f(t, v)| = |u - t|[1 - (u + t)/D] < |u - t|. \tag{7}$$

For $(x, y, z) \neq (x', y', z')$, (7) gives

$$|F(x, y, z) - F(x', y', z')| < (1/2)\sqrt{\Sigma(|y - y'| + |z - z'|)^2}$$
$$\leq \|(x, y, z) - (x', y', z')\| \tag{8}$$

where Σ stands for the cyclic sum and $\| \ \|$ denotes the Euclidean norm in R^3. Uniqueness follows immediately from (8).

REFERENCES

1. Nathan Altshiller Court, The problem of the three bisectors, *Scripta Mathematica*, XIX (June–September 1953), 218–219.
2. O. Bottema, A theorem of F. J. Van Den Berg (1833–92), *Nieuw Archief voor Wiskunde* (3), XXVI (1978), 161–171.
3. John Milnor, Analytic proofs of the 'Hairy Ball Theorem' and the Brouwer Fixed Point Theorem, this MONTHLY, 85 (1978), 521–524.

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