together make square numbers; the gnomons in the case of triangular numbers are the successive numbers 1, 2, 3, 4...; those for pentagonal numbers are the series 1, 4, 7, 10...(the common difference being 3), and so on. In general, the successive *gnomonic* numbers for any polygonal number, say of n sides, have $n-2$ for their common difference (Theon of Smyrna, p. 34, 13—15).

Geometrical Algebra.

We have already seen (cf. part of the note on I. 47 and the above note on the *gnomon*) how the Pythagoreans and later Greek mathematicians exhibited different kinds of numbers as forming different geometrical figures. Thus, says Theon of Smyrna (p. 36, 6—11), "plane numbers, triangular, square and solid numbers, and the rest, are not so called independently (κυρίως) but in virtue of their similarity to the areas which they measure; for 4, since it measures a square area, is called square by adaptation from it, and 6 is called oblong for the same reason." A "plane number" is similarly described as a number obtained by multiplying two numbers together, which two numbers are sometimes spoken of as "sides," sometimes as the "length" and "breadth" respectively, of the number which is their product.

The *product* of two numbers was thus represented geometrically by the *rectangle* contained by the straight lines representing the two numbers respectively. It only needed the discovery of incommensurable or irrational straight lines in order to represent geometrically by a rectangle the product of any two quantities whatever, rational or irrational; and it was possible to advance from a geometrical arithmetic to a geometrical *algebra*, which indeed by Euclid's time (and probably long before) had reached such a stage of development that it could solve the same problems as our algebra so far as they do not involve the manipulation of expressions of a degree higher than the second. In order to make the geometrical algebra so generally effective, the theory of proportions was essential. Thus, suppose that x, y, z etc. are quantities which can be represented by straight lines, while α, β, γ etc. are coefficients which can be expressed by ratios between straight lines. We can then by means of Book VI. find a single straight line d such that

$$\alpha x + \beta y + \gamma z + \dots = d.$$

To solve the simple equation in its general form

$$\alpha x + a = b,$$

where α represents any ratio between straight lines, also requires recourse to the sixth Book, though, e.g., if α is $\frac{1}{2}$ or $\frac{1}{3}$ or any submultiple of unity, or if α is 2, 4 or any power of 2, we should not require anything beyond Book I. for solving the equation. Similarly the general form of a quadratic equation requires Book VI. for its geometrical solution, though particular quadratic equations may be so solved by means of Book II. alone.

Besides enabling us to solve geometrically these particular quadratic equations, Book II. gives the geometrical proofs of a number of algebraical formulae. Thus the first ten propositions give the equivalent of the several identities

1. $a(b + c + d + \dots) = ab + ac + ad + \dots,$
2. $(a + b)a + (a + b)b = (a + b)^2,$
3. $(a + b)a = ab + a^2,$
4. $(a + b)^2 = a^2 + b^2 + 2ab,$

5. $ab + \left(\frac{a+b}{2} - b\right)^2 = \left(\frac{a+b}{2}\right)^2$,

or $(\alpha + \beta)(\alpha - \beta) + \beta^2 = \alpha^2$,

6. $(2a + b)b + a^2 = (a + b)^2$,

or $(\alpha + \beta)(\beta - \alpha) + \alpha^2 = \beta^2$,

7. $(a + b)^2 + a^2 = 2(a + b)a + b^2$,

or $\alpha^2 + \beta^2 = 2\alpha\beta + (\alpha - \beta)^2$,

8. $4(a + b)a + b^2 = \{(a + b) + a\}^2$,

or $4\alpha\beta + (\alpha - \beta)^2 = (\alpha + \beta)^2$,

9. $a^2 + b^2 = 2\left\{\left(\frac{a+b}{2}\right)^2 + \left(\frac{a+b}{2} - b\right)^2\right\}$,

or $(\alpha + \beta)^2 + (\alpha - \beta)^2 = 2(\alpha^2 + \beta^2)$,

10. $(2a + b)^2 + b^2 = 2\{a^2 + (a + b)^2\}$,

or $(\alpha + \beta)^2 + (\beta - \alpha)^2 = 2(\alpha^2 + \beta^2)$.

The form of these identities may of course be varied according to the different symbols which we may use to denote particular portions of the lines given in Euclid's figures. They are, for the most part, simple identities, but there is no reason to suppose that these were the only applications of the geometrical algebra that Euclid and his predecessors had been able to make. We may infer the very contrary from the fact that Apollonius in his *Conics* frequently states without proof much more complicated propositions of the kind.

It is important however to bear in mind that the whole procedure of Book II. is *geometrical*; rectangles and squares are shown in the figures, and the equality of certain combinations to other combinations is proved by those figures. We gather that this was the classical or standard method of proving such propositions, and that the *algebraical* method of proving them, with no figure except a line with points marked thereon, was a later introduction. Accordingly Eutocius' method of proving certain lemmas assumed by Apollonius (*Conics*, II. 23 and III. 29) probably represents more nearly than Pappus' proof of the same the point of view from which Apollonius regarded them.

It would appear that Heron was the first to adopt the *algebraical* method of demonstrating the propositions of Book II., beginning from the second, without figures, as consequences of the first proposition corresponding to

$$a(b + c + d) = ab + ac + ad.$$

According to an-Nairīzī (ed. Curtze, p. 89), Heron explains that it is not possible to prove II. 1 without drawing a number of lines (i.e. without actually drawing the rectangles), but that the following propositions up to II. 10 inclusive can be proved by merely drawing one line. He distinguishes two varieties of the method, one by *dissolutio*, the other by *compositio*, by which he seems to mean *splitting-up* of rectangles and squares, and *combination* of them into others. But in his proofs he sometimes combines the two varieties.

When he comes to II. 11, he says that it is not possible to do without a figure because the proposition is a problem, which accordingly requires an *operation* and therefore the drawing of a figure.

The algebraical method has been preferred to Euclid's by some English editors; but it should not find favour with those who wish to preserve the essential features of Greek geometry as presented by its greatest exponents, or to appreciate their point of view.

It may not be out of place to add a word with reference to the geometrical equivalent of the algebraical operations. The addition and subtraction of quantities represented in the geometrical algebra by lines is of course effected by producing the line to the required extent or cutting off a portion of it. The equivalent of multiplication is the construction of the rectangle of which the given lines are adjacent sides. The equivalent of the division of one quantity represented by a line by another quantity represented by a line is simply the statement of a *ratio* between lines on the principles of Books V. and VI. The division of a product of two quantities by a third is represented in the geometrical algebra by the finding of a rectangle with one side of a given length and equal to a given rectangle or square. This is the problem of *application of areas* solved in I. 44, 45. The addition and subtraction of products is, in the geometrical algebra, the addition and subtraction of rectangles or squares; the sum or difference can be transformed into a single rectangle by means of the *application of areas* to any line of given length, corresponding to the algebraical process of finding a common measure. Lastly, the extraction of the square root is, in the geometrical algebra, the finding of a square equal to a given rectangle, which is done in II. 14 with the help of I. 47.