Proof (sketch). One can give a direct argument, but it is simpler to derive these results from the functional equations stated earlier. To check (12.31) we appeal to a factorization of the Dedekind zeta-function

$$\zeta_K(s) = \zeta(s)L(s, \chi_D). \tag{12.33}$$

Hence, comparing the functional equations (12.20), (12.1) and (12.7), we deduce (12.31). To check (12.32) we appeal to the factorization

$$L(s, \psi \circ N) = L(s, \psi)L(s, \psi\chi_D). \tag{12.34}$$

Hence, comparing the functional equations (12.25) and (12.7), we deduce (12.32).

Now we are ready to present the main results. For clarity we state the imaginary field and the real field cases separately.

Theorem 12.5. *Let $K = \mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field with discriminant $D < 0$ and $\xi \pmod{\mathfrak{m}}$ a Hecke character such that*

$$\xi((a)) = \left(\frac{a}{|a|}\right)^u \quad \text{if } a \equiv 1 \pmod{\mathfrak{m}} \tag{12.35}$$

where u is a non-negative integer. Then

$$f(z) = \sum_{\mathfrak{a}} \xi(\mathfrak{a})(N\mathfrak{a})^{\frac{u}{2}} e(zN\mathfrak{a}) \in \mathcal{M}_k(\Gamma_0(N), \chi) \tag{12.36}$$

where $k = u + 1$, $N = |D|N\mathfrak{m}$ and $\chi \pmod{N}$ is the Dirichlet character given by

$$\chi(n) = \chi_D(n)\xi((n)) \quad \text{if } n \in \mathbb{Z}. \tag{12.37}$$

Moreover, f is a cusp form if $u > 0$.

Remark. We assume that $u \geqslant 0$ by conjugating ξ if necessary.

Proof (sketch). Note that $\xi \pmod{\mathfrak{m}}$ is not required to be primitive; nevertheless we only sketch the arguments for ξ primitive. Consider another function

$$g(z) = C \sum_{\mathfrak{a}} \bar{\xi}(\mathfrak{a})(N\mathfrak{a})^{\frac{u}{2}} e(zN\mathfrak{a}) \tag{12.38}$$

where $C = i^{-2u-1}W(\xi)(N\mathfrak{m})^{-1/2}$. By definition of f and g we have $L_f(s) = L\left(s-\frac{u}{2},\xi\right)$ and $L_g(s) = CL\left(s-\frac{u}{2},\bar{\xi}\right)$. To these we attach the factor $(\sqrt{N}/2\pi)^s\Gamma(s)$ to make the complete L-functions

$$\Lambda_f(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L\left(s-\frac{u}{2},\xi\right),$$

$$\Lambda_g(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)CL\left(s-\frac{u}{2},\bar{\xi}\right),$$

and verify by (12.26) that $\Lambda_f(s) = i^k\Lambda_g(k-s)$ (recall that $u = k-1$). Hence by the converse Theorem 7.3 it follows that

$$g = f_{|\omega_N}, \quad \omega_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix}.$$

Next let $\psi \pmod p$ be a primitive Dirichlet character of conductor $p \nmid N$. Then the twisted L-functions are given by $L_f(s,\psi) = L\left(s-\frac{u}{2},\xi\cdot\psi\circ N\right)$ and $L_g(s,\psi) = CL\left(s-\frac{u}{2},\bar{\xi}\cdot\psi\circ N\right)$. We set

$$\Lambda_f(s,\psi) = \left(\frac{p\sqrt{N}}{2\pi}\right)^s \Gamma(s)L\left(s-\frac{u}{2},\xi\cdot\psi\circ N\right)$$

$$\Lambda_g(s,\psi) = \left(\frac{p\sqrt{N}}{2\pi}\right)^s \Gamma(s)CL\left(s-\frac{u}{2},\bar{\xi}\cdot\psi\circ N\right)$$

and verify by (12.25) that

$$\Lambda_f(s,\psi) = i^k w(\psi)\Lambda_g(k-s,\bar{\psi})$$

with the appropriate root number

$$w(\psi) = \chi(p)\psi(N)\tau(\psi)^2p^{-1}$$

(to get exactly this number, use (12.37), (12.32), (12.23) and (12.22)). Hence by the converse Theorem 7.8 we obtain (12.36). If $u > 0$, then $L_f(s) = L\left(s-\frac{u}{2},\xi\right)$ converges absolutely in $\mathrm{Re}\, s > \frac{u}{2}+1$, where $\frac{u}{2}+1 = \frac{k+1}{2} < k$, so f is a cusp form.

Remark. If $\xi \pmod{\mathfrak{m}}$ is primitive, then f given by the Fourier series (12.36) and $g = f_{|\omega_N}$ given by the series (12.38) yield L-functions with adequate Euler products (i.e. of type (6.93)); therefore f is a newform, namely an eigenfunction of all the Hecke operators T_n^χ with eigenvalues

$$a_n = n^{\frac{k-1}{2}} \sum_{N\mathfrak{a}=n} \xi(\mathfrak{a}),$$

and f is also an eigenfunction of the operator $\bar{W} = KW$ with eigenvalue $\bar{C} = i^{2k-1}\bar{W}(\xi)(N\mathfrak{m})^{-1/2}$.

Theorem 12.6. *Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field with discriminant $D > 0$ and $\xi \pmod{\mathfrak{m}}$ a Hecke character such that*

$$\xi((a)) = \frac{a}{|a|} \quad \text{if } a \equiv 1 \pmod{\mathfrak{m}} \tag{12.39}$$

or

$$\xi((a)) = \frac{a'}{|a'|} \quad \text{if } a \equiv 1 \pmod{\mathfrak{m}} \tag{12.40}$$

where a' denotes the conjugate over $\mathbb{Q}$. Then

$$f(z) = \sum_{\mathfrak{a}} \xi(\mathfrak{a}) e(zN\mathfrak{a}) \in S_1(\Gamma_0(N), \chi) \tag{12.41}$$

where $N = DN\mathfrak{m}$ and the character $\chi \pmod{N}$ is defined as in (12.37).

Proof (hint). This follows from the converse Theorem 7.8 by arguments similar to those used in the proof of Theorem 12.5.

Note that a Hecke character on a real quadratic field always yields a cusp form of weight $k = 1$. If $\xi \pmod{\mathfrak{m}}$ is primitive, then f given by the Fourier series (12.41) is a newform with Hecke eigenvalues

$$a_n = \sum_{N\mathfrak{a}=n} \xi(\mathfrak{a}). \tag{12.42}$$

12.4. Class group L-functions reconsidered

Our arguments used for Theorem 12.5 and Theorem 12.6 were sketchy and the proofs were not really complete, since we appealed to Hecke's general results about analytic continuation and functional equations for L-functions attached to Grossencharacters. Our purpose was merely to illustrate an application of the Hecke-Weil converse theorems. Now we take a direct approach to show that the automorphic forms associated with L-functions of a quadratic field correspond to theta functions. For simplicity we confine the demonstration to characters of conductor $\mathfrak{m} = \mathcal{O}$, so such a character is primitive. Actually we only consider the class group characters. For an imaginary field this means $u = 0$ in (12.35). However if K is real and every unit of K has norm 1, then neither (12.39) nor (12.40) defines a function on principal ideals. In this case in order to speak of a class group character one needs a more subtle definition of ideal classes.

Let $K = \mathbb{Q}(\sqrt{D})$ be any quadratic field of discriminant D, positive or negative. Two ideals $\mathfrak{a}, \mathfrak{b} \in I$ are said to be equivalent in the narrow sense if

$$\mathfrak{b} = (\alpha)\mathfrak{a} \text{ with } \alpha \in K,\ N\alpha > 0.$$

This is a new concept only when K is real and all its units have norm 1. Here we may replace $N\alpha > 0$ by the condition $\alpha \succ 0$, which means $\alpha > 0$ and $\alpha' > 0$ in case of a real field and $\alpha \neq 0$ in case of an imaginary field. We put

$$P^+ = \{(a) : a \in K, a \succ 0\}.$$

Then $\mathcal{H}^+ = I/P^+$ is the class group of narrow classes and $h^+ = [I : P^+]$ is the narrow class number,

$$h^+ = \begin{cases} 2h & \text{if } K \text{ is real and } N\varepsilon = 1 \\ h & \text{otherwise} \end{cases}$$

where ε is the fundamental unit. By class character we mean a group homomorphism $\chi : \mathcal{H}^+ \to S^1$. With χ we associate the L-function

$$L_K(s,\chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})(N\mathfrak{a})^{-s} \tag{12.43}$$

where $\mathfrak{a}$ ranges over non-zero integral ideals. We split this series into narrow classes, getting

$$L_K(s,\chi) = \sum_{\mathcal{A}\in\mathcal{H}^+} \chi(\mathcal{A})\zeta_K(s,\mathcal{A})$$

where $\zeta_K(s,\mathcal{A})$ is the zeta-function of the class $\mathcal{A}$,

$$\zeta_K(s,\mathcal{A}) = \sum_{\mathfrak{a}\in\mathcal{A}} (N\mathfrak{a})^{-s}. \tag{12.44}$$

It turns out that every class zeta-function $\zeta_K(s,\mathcal{A})$ has meromorphic continuation over the s-plane and satisfies a functional equation of the same type. Hence the whole function $L_K(s,\chi)$ inherits these properties.

From now on we give further details only for an imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$ of discriminant $D < 0$. Let $w = \#U$ be the number of units of K. For every class $\mathcal{A} \in \mathcal{H}$ we introduce the theta function

$$\Theta_{\mathcal{A}}(z) = w^{-1} + \sum_{\mathfrak{a}\in\mathcal{A}} e(zN\mathfrak{a}), \tag{12.45}$$

and for any character $\chi \in \hat{\mathcal{H}}$ we put

$$\begin{aligned} f_\chi(z) &= \sum_{\mathcal{A}\in\mathcal{H}} \chi(\mathcal{A})\Theta_{\mathcal{A}}(z) \\ &= w^{-1}h\delta(\chi) + \sum_{\mathfrak{a}} \chi(\mathfrak{a})e(zN\mathfrak{a}) \end{aligned} \tag{12.46}$$

where $\delta(\chi) = 1$ if χ is trivial and $\delta(\chi) = 0$ otherwise. Every class $\mathcal{A}$ contains an integral primitive ideal (i.e. not divisible by a rational integer > 1). Every primitive ideal can be written as

$$\mathfrak{a} = \left[a, \frac{b+\sqrt{D}}{2}\right] \quad \text{with } a > 0,\ b^2 - 4ac = D,\ (a,b,c) = 1. \tag{12.47}$$

The above notation means $\mathfrak{a}$ is a free $\mathbb{Z}$-module,

$$\mathfrak{a} = a\mathbb{Z} + \frac{b+\sqrt{D}}{2}\mathbb{Z}.$$

Note that $\frac{b+\sqrt{D}}{2} \in \mathcal{O}$ and $a = N\mathfrak{a}$; indeed, we obtain

$$\begin{aligned} \mathfrak{a}\bar{\mathfrak{a}} &= \left[a, \frac{b+\sqrt{D}}{2}\right]\left[a, \frac{b-\sqrt{D}}{2}\right] = \left[a^2, a\frac{b+\sqrt{D}}{2}, a\frac{b-\sqrt{D}}{2}, c\right] \\ &= [a^2, ab, ac] = a[a,b,c] = a\mathcal{O}. \end{aligned}$$

With the generators of $\mathfrak{a}$ we associate the quadratic form

$$\varphi_{\mathcal{A}}(x) = ax_1^2 + bx_1x_2 + cx_2^2 = \frac{1}{2}A[x]$$

where $A = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$. This establishes a one-to-one correspondence between the ideal classes $\mathcal{A} \in \mathcal{H}$ and the equivalence classes of primitive binary quadratic forms $\varphi_{\mathcal{A}}$ of discriminant D. We choose $\sqrt{D} = i\sqrt{|D|}$ so that

$$z_{\mathfrak{a}} = \frac{b+\sqrt{D}}{2a} \in \mathbb{H}.$$

Then the inverse ideal $\mathfrak{a}^{-1}$ is a free $\mathbb{Z}$-module generated by 1 and $\bar{z}_{\mathfrak{a}}$,

$$\mathfrak{a}^{-1} = [1, \bar{z}_{\mathfrak{a}}] = \mathbb{Z} + \frac{b-\sqrt{D}}{2a}\mathbb{Z}.$$

Now, given a class $\mathcal{A}$ which contains $\mathfrak{a}$, we can write

$$\Theta_{\mathcal{A}}(z) = w^{-1} + \sum_{\mathfrak{b}\sim\mathfrak{a}} e(zN\mathfrak{b}).$$

Here the equivalence $\mathfrak{b} \sim \mathfrak{a}$ means $\mathfrak{b} = (\alpha)\mathfrak{a}$ with $\alpha \in \mathfrak{a}^{-1}, \alpha \neq 0$, i.e. $\alpha = m + n\bar{z}_{\mathfrak{a}}$ with $m, n \in \mathbb{Z}$, not both zero. As m, n range over the integers every ideal $\mathfrak{b} \sim \mathfrak{a}$ is covered exactly w times. Moreover we have $N\mathfrak{b} = |\alpha|^2 a = am^2 + bmn + cn^2$, whence

$$\Theta_{\mathcal{A}}(z) = w^{-1} \sum_{(m,n)\in\mathbb{Z}^2} e(\varphi_A(m,n)z). \tag{12.48}$$

This is indeed the theta function associated with the matrix $A = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$. Note that $-DA^{-1} = \begin{pmatrix} 2c & -b \\ -b & 2a \end{pmatrix}$. Therefore by Theorem 10.9

$$\Theta_{\mathcal{A}} \in \mathcal{M}_1(\Gamma_0(|D|), \chi_D) \tag{12.49}$$

(observe the consistency condition $\chi_D(-1) = -1$). Furthermore, by the Jacobi inversion formula (10.10) we get

$$\Theta_{\mathcal{A}}(z) = |D|^{-1/2} iz^{-1} \Theta_{\mathcal{A}^{-1}}(-1/|D|z) \tag{12.50}$$

where $\mathcal{A}^{-1}$ denotes the inverse class to $\mathcal{A}$. Averaging over the ideal classes $\mathcal{A} \in \mathcal{H}$, each one weighted by $\chi(\mathcal{A})$, we extend the above automorphy relations to f_χ. These show that

$$f_\chi \in \mathcal{M}_1(\Gamma_0(|D|), \chi_D) \tag{12.51}$$

$$f_{\chi|\left(\begin{smallmatrix} & -1 \\ |D| & \end{smallmatrix}\right)} = -if_\chi \tag{12.52}$$

respectively (note that $f_\chi = f_{\bar\chi}$).

Put

$$\Lambda_K(s,\chi) = \left(\frac{\sqrt{|D|}}{2\pi}\right)^s \Gamma(s) L_K(s,\chi), \tag{12.53}$$

and observe that the functions $f = f_\chi$ and $g = f_{\chi|\left(\begin{smallmatrix} & -1 \\ |D| & \end{smallmatrix}\right)} = -if_\chi$ have the Fourier series (7.20) and (7.21) with constant terms $a_0 = w^{-1}h\delta(\chi)$ and $b_0 = -iw^{-1}h\delta(\chi)$ respectively (to compute the constant terms, use the asymptotics for $s \to 1$ from the end of Section 7.2). Therefore (A_2) of Theorem 7.3 asserts that the function

$$\Lambda_K(s,\chi) + w^{-1}h\delta(\chi)(s(1-s))^{-1} \tag{12.54}$$

is entire and bounded in vertical strips, and $\Lambda_K(s,\chi)$ satisfies the functional equation

$$\Lambda_K(s,\chi) = \Lambda_K(1-s,\chi). \tag{12.55}$$

Actually we have the following integral representation:

$$\Lambda_K(s,\chi) + w^{-1}h\delta(\chi)(s(1-s))^{-1}$$
$$= \int_1^\infty (y^{s-1} + y^{-s})\Big(\sum_{\mathfrak{a}} \chi(\mathfrak{a}) \exp(-2\pi y N\mathfrak{a}/\sqrt{|D|})\Big)dy$$

from which the above assertions follow at once. As a by-product we derive from (12.53), (12.54) and (12.33) the Dirichlet class number formula (examine the residue at $s = 1$):

$$h = \frac{w\sqrt{|D|}}{2\pi} L(1, \chi_D). \tag{12.56}$$

If $K = \mathbb{Q}(\sqrt{D})$ is real ($D > 0$), the above analysis fails since the group of units is infinite while the quadratic forms corresponding to ideal classes are indefinite. The latter topics lie beyond the scope of this book, so we don't proceed beyond what has been stated about $L_K(s,\chi)$ in Theorem 12.2 (if χ is trivial) and Theorem 12.3 (if χ is non-trivial).

12.5. L-functions for genus characters

Next we shall examine $L_K(s,\chi)$ closely for special class group characters (for either imaginary or real quadratic fields), the genus characters. The genus theory was created by Gauss in the context of binary quadratic forms. Here we recall its content in terms of ideals.

A discriminant of a quadratic field is said to be a prime discriminant if it has only one prime factor, so it must be one of the following type:

$$-4, \ \pm 8, \ \pm p \equiv 1 \pmod 4. \tag{12.57}$$

The product of coprime discriminants is again a discriminant. Every discriminant can be written uniquely as a product of prime discriminants, say $D = P_1 \dots P_t$. Hence D can be arranged as a product of two discriminants

$$D = D_1 D_2 \tag{12.58}$$

in 2^{t-1} distinct ways if we allow interchanging D_1 with D_2 (here and hereafter, t denotes the number of distinct prime factors of D). For any such decomposition we define a character χ_{D_1,D_2} on ideals by setting

$$\chi_{D_1,D_2}(\mathfrak{p}) = \begin{cases} \chi_{D_1}(N\mathfrak{p}) & \text{if } \mathfrak{p} \nmid D_1 \\ \chi_{D_2}(N\mathfrak{p}) & \text{if } \mathfrak{p} \nmid D_2 \end{cases} \tag{12.59}$$

(recall that $\chi_d(n) = \left(\frac{d}{n}\right)$ is the Kronecker symbol). This is well defined on prime ideals because

$$\chi_D(N\mathfrak{a}) = 1 \quad \text{if } (\mathfrak{a}, D) = 1, \tag{12.60}$$

and we extend χ_{D_1,D_2} to all fractional ideals by multiplicativity. Therefore

$$\chi_{D_1,D_2} : I \to \{\pm 1\}$$

so χ_{D_1,D_2} has order two, except for the trivial character which corresponds to the trivial factorizations $D = D \cdot 1 = 1 \cdot D$. Every such χ_{D_1,D_2} is called the genus character of discriminant D; these are different for distinct factorizations (12.58), so we have exactly 2^{t-1} genus characters.

The genus characters are the narrow class group characters, $\chi_{D_1,D_2} \in \mathcal{H}^+$, i.e.

$$\chi_{D_1,D_2}(\mathfrak{a}) = 1 \quad \text{if } \mathfrak{a} = (\alpha), \alpha \succ 0. \tag{12.61}$$

Theorem 12.7 (Kronecker). *The L-function of $K = \mathbb{Q}(\sqrt{D})$ associated with the genus character χ_{D_1,D_2} factors into the Dirichlet L-functions,*

$$L_K(s, \chi_{D_1,D_2}) = L(s, \chi_{D_1})L(s, \chi_{D_2}). \tag{12.62}$$

Proof (hint). For the trivial character ($D_1 = 1$ or $D_2 = 1$) this is the factorization (12.33) for the Dedekind zeta-function. In general one can verify the Kronecker factorization (12.62) easily by examining the local factors in the corresponding Euler products and using the law of factorization of primes in K expressed in terms of values $\chi_D(p) = 0, \pm 1$.

One shows that every real character of $\mathcal{H}^+$ is a genus character, and they all form a group of order 2^{t-1} which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{t-1}$.

Now let us describe the dual side of genus characters. We say that two non-zero ideals $\mathfrak{a}, \mathfrak{b} \in I$ are in the same genus if

$$\chi(\mathfrak{a}) = \chi(\mathfrak{b}) \quad \text{for all genus characters.} \tag{12.63}$$

The ideals with $\chi(\mathfrak{a}) = 1$ for all genus characters form the principal genus (a subgroup of I). The same definitions apply to the narrow classes. We denote the subgroup of principal genus classes by

$$\mathcal{G} = \{\mathcal{A} \in \mathcal{H}^+ : \chi(\mathcal{A}) = 1 \text{ for all genus characters}\}. \tag{12.64}$$

The factor group $\mathcal{F} = \mathcal{H}^+/\mathcal{G}$ is called the genus group. By duality $\mathcal{F}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{t-1}$.