Pas de titre

Emmanuel Vieillard-Baron[1], Alain Soyeur[2], and François Capaces[3]

[1]Enseignant en CPGE, Lycée Kléber, Strasbourg
[2]Enseignant en CPGE, Lycée Pierre de Fermat, Toulouse
[3], ,

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Exercice 0.1 ★ ***Pas de titre***

Déterminer un équivalent des fonctions suivantes au voisinage du point indiqué :

1. $f(x) = \dfrac{e^x - \sqrt{1+x}}{(x^2+1)(x+3)}$ *en* $x = 0$.
2. $f(x) = \left(\dfrac{\operatorname{sh} x}{x}\right)^{\sin x} - \left(\dfrac{\sin x}{x}\right)^{\operatorname{sh} x}$ *en* $x = 0$.
3. $f(x) = \dfrac{x^2 + \cos x - \operatorname{ch} x}{\sqrt{x}}$ *en* $x = 0$.
4. $f(x) = \sqrt{x} - \sqrt{\sin x}$ *en* $x = 0^+$.
5. $f(x) = \operatorname{sh}(\sin x) - \sin(\operatorname{sh} x)$ *en* $x = 0$.
6. $f(x) = \arctan(2x) - 2\arctan(x)$ *en* $x = 0$.
7. $f(x) = \arctan\sin x - \sin\arctan x$ *en* $x = 0$.
8. $f(x) = e^{\frac{1}{x}} - e^{\frac{1}{x+1}}$ *en* $x = +\infty$.

Solution :

1. *On a* $e^x - \sqrt{1+x} = (1+x) - \left(1 + \dfrac{x}{2}\right) + \underset{x\to 0}{o}(x) = \dfrac{x}{2} + \underset{x\to 0}{o}(x)$ *et* $\left(x^2+1\right)(x+3) \underset{x\to 0}{\sim} 3$. *Donc*
$$\frac{e^x - \sqrt{1+x}}{(x^2+1)(x+3)} \underset{x\to 0}{\sim} \boxed{\frac{x}{6}}.$$
2. *On a :* $\left(\dfrac{\operatorname{sh} x}{x}\right)^{\sin x} = 1 + \dfrac{x^3}{6} + \underset{x\to 0}{o}\left(x^3\right)$ *et* $\left(\dfrac{\sin x}{x}\right)^{\operatorname{sh} x} = 1 - \dfrac{x^3}{6} + \underset{x\to 0}{o}\left(x^3\right)$ *donc :*
$$\begin{aligned}\left(\frac{\operatorname{sh} x}{x}\right)^{\sin x} - \left(\frac{\sin x}{x}\right)^{\operatorname{sh} x} &= \frac{1}{3}x^3 + \underset{x\to 0}{o}\left(x^3\right)\\ &\underset{x\to 0}{\sim} \boxed{\frac{1}{3}x^3}\end{aligned}$$
3. $$\begin{aligned}\frac{x^2 + \cos x - \operatorname{ch} x}{\sqrt{x}} &= \frac{-\frac{1}{360}x^6 + \underset{x\to 0}{o}\left(x^6\right)}{\sqrt{x}}\\ &\underset{x\to 0}{\sim} \boxed{-\frac{x^{\frac{11}{2}}}{360}}\end{aligned}$$

4.

$$\begin{aligned}\sqrt{x}-\sqrt{\sin x} &= \sqrt{x}-\sqrt{x-\frac{x^3}{6}+\underset{x\to0}{o}\left(x^3\right)}\\ &= \sqrt{x}\left(1-\sqrt{1-\frac{x^2}{6}+\underset{x\to0}{o}\left(x^2\right)}\right)\\ &= \sqrt{x}\left(\frac{1}{12}x^2+\underset{x\to0}{o}\left(x^2\right)\right)\\ &\underset{x\to0}{\sim} \boxed{\frac{x^{\frac{5}{2}}}{12}}\end{aligned}$$

car $\dfrac{\sin x}{x}\xrightarrow[x\to0]{}1$.

5. *On a :* $\operatorname{sh}(\sin x)=x-\dfrac{x^5}{15}+\dfrac{x^7}{90}+\underset{x\to0}{o}\left(x^7\right)$ *et* $\sin(\operatorname{sh}x)=x-\dfrac{x^5}{15}-\dfrac{x^7}{90}+\underset{x\to0}{o}\left(x^7\right)$ *donc :*

$$\begin{aligned}\operatorname{sh}(\sin x)-\sin(\operatorname{sh}x) &= \frac{x^7}{45}+\underset{x\to0}{o}\left(x^7\right)\\ &\underset{x\to0}{\sim}\boxed{\frac{x^7}{45}}\end{aligned}$$

6. *On a :* $\arctan x = x-\dfrac{x^3}{3}+\underset{x\to0}{o}\left(x^3\right)$ *et donc :* $\arctan 2x = 2x-\dfrac{8x^3}{3}+\underset{x\to0}{o}\left(x^3\right)$ *ce qui amène :*

$$\begin{aligned}\arctan(2x)-2\arctan(x) &= -2x^3+\underset{x\to0}{o}\left(x^3\right)\\ &\underset{x\to0}{\sim}\boxed{-2x^3}\end{aligned}$$

7. *On a* $\arctan(\sin(x))=x-\dfrac{1}{2}x^3+\dfrac{3}{8}x^5-\dfrac{83}{240}x^7+\underset{x\to0}{o}\left(x^7\right)$ *et* $\sin(\arctan(x))=x-\dfrac{1}{2}x^3+\dfrac{3}{8}x^5-\dfrac{5}{16}x^7+\underset{x\to0}{o}\left(x^7\right)$ *donc :*

$$\begin{aligned}\arctan\sin x-\sin\arctan x &= -\frac{x^7}{30}+\underset{x\to0}{o}\left(x^7\right)\\ &\underset{x\to0}{\sim}\boxed{-\frac{x^7}{30}}\end{aligned}$$

8. *Posons* $X=\dfrac{1}{x}$

$$\begin{aligned}e^{\frac{1}{x}}-e^{\frac{1}{x+1}} &= e^X-e^{\frac{X}{1+X}}\\ &= e^X-e^{X\left(1+X+\underset{X\to0}{o}(X)\right)}\\ &= e^X-e^{X+X^2+\underset{X\to0}{o}\left(X^2\right)}\\ &= 1+X+\frac{X^2}{2}-\left(1+X-\frac{X^2}{2}\right)+\underset{X\to0}{o}\left(X^2\right)\\ &= X^2+\underset{X\to0}{o}\left(X^2\right)\\ &\underset{X\to0}{\sim} X^2\\ &\underset{x\to+\infty}{\sim}\boxed{\frac{1}{x^2}}\end{aligned}$$

Références