

Pas de titre

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Exercice 0.1 ★ Pas de titre
Déterminer, en utilisant des développements limités, les limites suivantes :

$$1. \lim_{x \rightarrow 1} \frac{\ln(2x^2 - 1)}{\tan(x-1)}$$

$$4. \lim_{x \rightarrow 0} \frac{1}{x^3} - \frac{1}{\sin^3 x}$$

$$2. \lim_{x \rightarrow +\infty} x - x^2 \ln\left(1 + \frac{1}{x}\right)$$

$$5. \lim_{x \rightarrow 0} \frac{1}{x} - \frac{1}{\ln(1+x)}$$

$$3. \lim_{x \rightarrow 0} \frac{x - \arctan x}{\sin^3 x}$$

$$6. \lim_{x \rightarrow 0} \left(\frac{1^x + 2^x + \dots + n^x}{n} \right)^{\frac{1}{x}}$$

Solution :

1.

$$\begin{aligned} \frac{\ln(2x^2 - 1)}{\tan(x-1)} &\xrightarrow[X=x-1]{} \frac{\ln(1+4X+2X^2)}{\tan X} \\ &= \frac{4X + \underset{X \rightarrow 0}{o}(X)}{X + \underset{X \rightarrow 0}{o}(X)} \\ &= \frac{4 + \underset{X \rightarrow 0}{o}(1)}{1 + \underset{X \rightarrow 0}{o}(1)} \\ &\xrightarrow[X \rightarrow 0]{} \boxed{4} \end{aligned}$$

2.

$$\begin{aligned} x - x^2 \ln\left(1 + \frac{1}{x}\right) &\xrightarrow[X=\frac{1}{x}]{} \frac{1}{X} - \frac{1}{X^2} \ln(1+X) \\ &= \frac{1}{X} - \frac{1}{X^2} \left(X - \frac{X^2}{2} + \underset{X \rightarrow 0}{o}(X^2) \right) \\ &= \frac{1}{2} + \underset{X \rightarrow 0}{o}(X) \end{aligned}$$

$\xrightarrow[X \rightarrow 0]{}$

$\boxed{\frac{1}{2}}$

3.

$$\begin{aligned}
 \frac{x - \arctan x}{\sin^3 x} &\underset{x \rightarrow 0}{\sim} \frac{x - \arctan x}{x^3} \\
 &= \frac{\frac{1}{3}x^3 + o_{x \rightarrow 0}(x^3)}{x^3} \\
 &= \frac{\frac{1}{3}}{1} + o_{x \rightarrow 0}(1) \\
 &\xrightarrow[x \rightarrow 0]{\hspace{1cm}} \boxed{\frac{1}{3}}
 \end{aligned}$$

4.

$$\begin{aligned}
 \frac{1}{x^3} - \frac{1}{\sin^3 x} &= \frac{\sin^3 x - x^3}{x^3 \sin^3 x} \\
 &\underset{x \rightarrow 0}{\sim} \frac{\sin^3 x - x^3}{x^6} \\
 &= \frac{-\frac{x^5}{2} + o_{x \rightarrow 0}(x^6)}{x^6} \\
 &= -\frac{1}{2x} + o_{x \rightarrow 0}(1)
 \end{aligned}$$

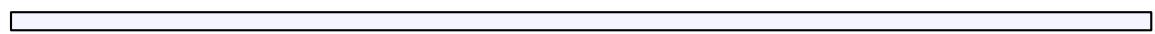
et $\boxed{\lim_{x \rightarrow 0^+} \frac{1}{x^3} - \frac{1}{\sin^3 x} = -\infty}$ tandis que $\boxed{\lim_{x \rightarrow 0^-} \frac{1}{x^3} - \frac{1}{\sin^3 x} = +\infty}$

5.

$$\begin{aligned}
 \frac{1}{x} - \frac{1}{\ln(1+x)} &= \frac{\ln(1+x) - x}{x \ln(1+x)} \\
 &\underset{x \rightarrow 0}{\sim} \frac{\ln(1+x) - x}{x^2} \\
 &= \frac{-\frac{x^2}{2} + o_{x \rightarrow 0}(x^2)}{x^2} \\
 &= -\frac{1}{2} + o_{x \rightarrow 0}(1) \\
 &\xrightarrow[x \rightarrow 0]{\hspace{1cm}} -\frac{1}{2}
 \end{aligned}$$

6.

$$\begin{aligned}
 \left(\frac{1^x + 2^x + \dots + n^x}{n} \right)^{\frac{1}{x}} &= \left(\frac{(1 + x \ln 1) + (1 + x \ln 2) + \dots + (1 + x \ln n) + o_{x \rightarrow 0}(x)}{n} \right)^{\frac{1}{x}} \\
 &= \left(\frac{n + (\ln 1 + \dots + \ln n)x + o_{x \rightarrow 0}(x)}{n} \right)^{\frac{1}{x}} \\
 &= e^{\frac{\ln(n!^{\frac{1}{n}})x + o_{x \rightarrow 0}(x)}{x}} \\
 &= e^{\frac{\ln(n!^{\frac{1}{n}})x + o_{x \rightarrow 0}(x)}{x}} \\
 &= e^{\left(\ln(n!^{\frac{1}{n}}) + o_{x \rightarrow 0}(1) \right)} \\
 &\xrightarrow[x \rightarrow 0]{\hspace{1cm}} e^{\ln(n!^{\frac{1}{n}})} = \boxed{n!^{\frac{1}{n}}}
 \end{aligned}$$



Références