Pas de titre

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7 avril 2023

Exercice 0.1 ★ ***Pas de titre***

Calculer les intégrales suivantes en utilisant un changement de variable adéquat :

1. $\int_0^1 \frac{1}{1+e^x}\,dx$
2. $\int_0^1 x^2\sqrt{1-x^2}\,dx$
3. $\int_0^{\frac{1}{2}} \sqrt{\frac{\arcsin x}{1-x^2}}\,dx$
4. $\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x\sqrt{\tan x}}\,dx$
5. $\int_1^4 \frac{1}{x+\sqrt{x}}\,dx$
6. $\int_0^1 \frac{1}{1+x+x^2}\,dx$

Solution :

1. *On pose* $\begin{cases} u = e^x \\ du = e^x dx = u dx \end{cases}$, *on obtient :* $\int_0^1 \frac{1}{1+e^x}\,dx = \int_1^e \frac{1}{u(1+u)}\,du = \int_1^e \frac{1}{u} - \frac{1}{u+1}\,du = \Big[\ln u - \ln(1+u)\Big]_1^e = \boxed{\ln 2 - \ln(e+1) + 1}$.
2. *On pose* $\begin{cases} u = \cos x \\ du = -\sin x dx \end{cases}$, *on obtient :* $\int_0^1 x^2\sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \cos^2 u \sin^2 u\,du = \int_0^{\frac{\pi}{2}} \frac{1-\cos 4u}{8}\,du = \boxed{\frac{\pi}{16}}$.
3. *On pose* $\begin{cases} u = \arcsin x \\ du = \frac{dx}{\sqrt{1-x^2}} \end{cases}$, *on obtient :* $\int_0^{\frac{1}{2}} \sqrt{\frac{\arcsin x}{1-x^2}}\,dx = \int_0^{\frac{\pi}{6}} \sqrt{\frac{u}{1-\sin^2 u}}\sqrt{1-\sin^2 u}\,du = \int_0^{\frac{\pi}{6}} \sqrt{u}\,du = \boxed{\frac{\pi^{\frac{3}{2}}\sqrt{6}}{54}}$.
4. *On pose* $\begin{cases} u = \sqrt{\tan(x)} \\ du = \frac{1}{2\cos^2 x\sqrt{\tan x}} dx \end{cases}$, *on obtient :* $\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x\sqrt{\tan x}}\,dx = \int_0^1 2\,du = \boxed{2}$.

5. *On pose* $\begin{cases} u = \sqrt{x} \\ du = \dfrac{dx}{2\sqrt{x}} \end{cases}$ *, on obtient :* $\int_1^4 \dfrac{1}{x+\sqrt{x}}\,dx = \int_1^2 \dfrac{2}{1+u}\,du = \Big[2\ln(1+u)\Big]_1^2 = \boxed{2\ln\left(\dfrac{3}{2}\right)}$.

6. *On a :* $\int_0^1 \dfrac{1}{1+x+x^2}\,dx = \int_0^1 \dfrac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \int_0^1 \dfrac{4}{3}\dfrac{1}{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1}\,dx$ *On pose* $\begin{cases} u = \dfrac{2}{\sqrt{3}}\left(x+\dfrac{1}{2}\right) \\ du = \dfrac{2}{\sqrt{3}}dx \end{cases}$ *, on obtient :* $\int_0^1 \dfrac{1}{1+x+x^2}\,dx = \dfrac{2\sqrt{3}}{3}\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \dfrac{1}{1+u^2}\,du = \frac{2\sqrt{3}}{3}\Big[\arctan u\Big]_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} = \boxed{\dfrac{\pi\sqrt{3}}{9}}$.

Références