Pas de titre

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Exercice 0.1 ★ **_Pas de titre_**

Déterminer les primitives suivantes en utilisant le changement de variable précisé :

1. $\int_1^e \frac{\ln x}{x}\,dx$ *en posant* $u = \ln x$
2. $\int_0^1 \frac{x^3}{\sqrt{x+1}}\,dx$ *en posant* $u = \sqrt{x+1}$.
3. $\int_0^{\frac{\pi}{2}} \sin x \cos^2 x\,dx$ *en posant* $u = \cos x$.
4. $\int_0^{\frac{\pi}{4}} \tan^4 x\,dx$ *en posant* $u = \tan x$.
5. $\int_0^1 \sqrt{1-x^2}\,dx$ *en posant* $x = \cos u$.
6. $\int_1^e \frac{\ln x}{\sqrt{x}}\,dx$ *en posant* $u = \sqrt{x}$.

Solution :

1. *On pose* $u = \ln x$. u *est bien* $\mathcal{C}^1$ *et* $\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \boxed{\frac{1}{2}}$.

   Point n'est besoin de changement de variable puisque
   $\displaystyle\int \frac{\ln x}{x}\,dx = \frac{1}{2}\ln^2 x + C.$
2. *On pose* $\begin{cases} u = \sqrt{x+1} \\ du = \dfrac{dx}{2\sqrt{x+1}} \end{cases}$ *et* $x = u^2 - 1$. *On a :* $\int_0^1 \frac{x^3}{\sqrt{x+1}}\,dx = \int_1^{\sqrt{2}} \frac{\left(u^2-1\right)^3}{2}\,du = \boxed{\frac{16}{35} - \frac{9}{35}\sqrt{2}}$
3. *On pose* $\begin{cases} u = \cos x \\ du = -\sin x dx \end{cases}$, *on obtient :* $\int_0^{\frac{\pi}{2}} \sin x \cos^2 x\,dx = \int_0^1 u^2\,du = \boxed{\frac{1}{3}}$.
4. *On pose* $\begin{cases} u = \tan x \\ du = \left(1+\tan^2 x\right)dx = \left(1+u^2\right)dx \end{cases}$, *on obtient :* $\int_0^{\frac{\pi}{4}} \tan^4 x\,dx = \int_0^1 \frac{u^4}{1+u^2}\,du = \int_0^1 \left(\frac{u^4-1}{1+u^2} + \frac{1}{1+u^2}\right)du = \int_0^1 \left(u^2-1\right)\,du + \int_0^1 \frac{1}{1+u^2}\,du = \left[\frac{u^3}{3} - u + \arctan u\right]_0^1 = \boxed{-\frac{2}{3} + \frac{\pi}{4}}$.

5. *On pose* $\begin{cases} x = \cos u \\ dx = -\sin u du \end{cases}$, *on obtient :* $\int_0^1 \sqrt{1-x^2}\, dx = -\int_{\frac{\pi}{2}}^0 \sqrt{1-\cos^2 u} \sin u\, du = \int_0^{\frac{\pi}{2}} \sin^2 u\, du = \int_0^{\frac{\pi}{2}} \frac{1-\cos 2u}{2}\, du = \boxed{\frac{\pi}{4}}$.

6. *On pose* $\begin{cases} u = \sqrt{x} \\ du = \dfrac{dx}{2\sqrt{x}} \end{cases}$, *on obtient :* $\int_1^e \frac{\ln x}{\sqrt{x}}\, dx = \int_1^{\sqrt{e}} 2 \ln u^2\, du = \int_1^{\sqrt{e}} 4 \ln u\, du = 4\Big[u \ln u - u\Big]_1^{\sqrt{e}} = \boxed{4 - 2\sqrt{e}}$.

Références