Pas de titre

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22 septembre 2021

Exercice 0.1 ***Pas de titre***

Calculer :

1. $\int_2^3 \frac{1}{x(x-1)}\,dx$
2. $\int_2^3 \frac{2x+1}{x^2-1}\,dx$
3. $\int_0^1 \frac{x^2-1}{x^2+4x-5}\,dx$
4. $\int_0^{\frac{1}{2}} \frac{x+1}{\left(x^2+1\right)(x-2)}\,dx$
5. $\int_0^1 \frac{1}{\left(x^2+2x+5\right)(x+2)}\,dx$
6. $\int_0^2 \frac{x}{\left(x^2+x+1\right)(x+1)}\,dx$
7. $\int_0^1 \frac{1}{\left(x^2+x+1\right)^2}\,dx$
8. $\int_0^{-1} \frac{1}{\left(x^2-3x+2\right)^2}\,dx$
9. $\int_0^1 \frac{x-1}{\left(x^2+1\right)^2(x+2)}\,dx$

Solution :

1. $\frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{x-1}$ *donc* $\int_2^3 \frac{1}{x(x-1)}\,dx = \Big[-\ln|x| + \ln|x-1|\Big]_2^3 = \ln\left(\frac{4}{3}\right)$
2. $\int_2^3 \frac{2x+1}{x^2-1}\,dx = \frac{3}{2}\int_2^3 \frac{1}{x-1}\,dx + \int_1^2 \frac{1}{x+1}\,dx = \frac{1}{2}\Big[3\ln|x-1| + \ln|x+1|\Big]_2^3 = \frac{1}{2}\left(3\ln 2 + \ln 4 - \ln 3\right)$
3. $\int_0^1 \frac{x^2-1}{x^2+4x-5}\,dx = \int_0^1 \left(\frac{x+1}{x+5}\right)\,dx = \int_0^1 1 - \frac{4}{x+5}\,dx = \Big[x - 4\ln|x+5|\Big]_0^1 = 1 - 4\ln 6 + 4\ln 5.$
4. $\int_0^{\frac{1}{2}} \frac{x+1}{\left(x^2+1\right)(x-2)}\,dx = \frac{3}{5}\int_0^{\frac{1}{2}} \frac{1}{x-2}\,dx - \frac{3}{5}\int_0^{\frac{1}{2}} \frac{x+\frac{1}{2}}{x^2+1}\,dx = \Big[\frac{3}{5}\ln|x-2|\Big]_0^{\frac{1}{2}} - \frac{3}{5}\int_0^{\frac{1}{2}} \frac{x+\frac{1}{2}}{x^2+1}\,dx = \frac{3}{5}\ln\left(\frac{3}{4}\right) - \frac{3}{5}\int_0^{\frac{1}{2}} \frac{x}{x^2+1}\,dx - \frac{1}{5}\int_0^{\frac{1}{2}} \frac{1}{x^2+1}\,dx = \frac{3}{5}\ln\left(\frac{3}{4}\right) + \Big[-\frac{3}{10}\ln\left|x^2+1\right| - \frac{1}{5}\arctan x\Big]_0^{\frac{1}{2}} = \frac{3}{5}\ln(\frac{3}{4}) - \frac{3}{10}\ln(\frac{5}{4}) - \frac{1}{5}\arctan(\frac{1}{2})$
5. *On écrit la décomposition a priori* $\dfrac{1}{(x+2)(x^2+2x+5)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+2x+5}$.
   En multipliant les deux membres par $x+2$ *et en faisant* $x=-2$ *on trouve* $A = \dfrac{1}{4-4+5} = \dfrac{1}{5}$.
   En multipliant les deux membres par x^2+2x+5 *et en faisant* $x=-1+2i$ *on trouve*

$B(-1+2i)+C = \dfrac{1}{-1+2i+2} = \dfrac{1-2i}{5}$. *D'où* $B = -\dfrac{1}{5}$ *et* $C = 0$.

$$\begin{aligned}\int_0^1 \frac{dx}{(x+2)(x^2+2x+5)} &= \frac{1}{5}\int_0^1 \frac{dx}{x+2} - \frac{1}{5}\int_0^1 \frac{dx}{x^2+2x+5}\\ &= \frac{1}{10}\left[2\ln(x+2) - \ln(x^2+2x+5)\right]_0^1\\ &= \frac{1}{10}\left(2\ln 3 - 2\ln 2 - \ln 8 + \ln 5\right)\\ &= \frac{1}{10}\ln\left(\frac{45}{32}\right).\end{aligned}$$

6. *On écrit la décomposition a priori* $\dfrac{x}{(x+1)(x^2+x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+x+1}$.
   En multipliant les deux membres par $x+1$ *et en faisant* $x=-1$ *on trouve* $A = \dfrac{-1}{1-1+1} = -1$.
   En multipliant les deux membres par x^2+x+1 *et en faisant* $x=j$ *on trouve* $Bj+C = \dfrac{j}{j+1} = \dfrac{j(j^2+1)}{(j+1)(j^2+1)} = 1+j$. *D'où* $B=1$ *et* $C=1$.

$$\begin{aligned}\int_0^2 \frac{x\,dx}{(x+1)(x^2+x+1)} &= -\int_0^2 \frac{dx}{x+1} + \frac{1}{2}\int_0^2 \frac{2x+1}{x^2+x+1}\,dx\frac{1}{2}\int_0^2 \frac{dx}{x^2+x+1}\\ &= \left[-\ln(x+1) + \frac{1}{2}\ln(x^2+x+1)\right]_0^2 + \frac{1}{2}\int_0^2 \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}\\ &= -\ln 3 + \frac{1}{2}\ln 7 + \frac{1}{2}\times\frac{4}{3}\int_0^2 \frac{dx}{(\frac{2x+1}{\sqrt{3}})^2+1}\end{aligned}$$

On pose $u = \frac{2x+1}{\sqrt{3}}$, $du = \dfrac{2\,dx}{\sqrt{3}}$,

$$\begin{aligned}\frac{1}{2}\times\frac{4}{3}\int_0^2 \frac{dx}{(\frac{2x+1}{\sqrt{3}})^2+1} &= \frac{\sqrt{3}}{6}\int_{\frac{1}{\sqrt{3}}}^{\frac{5}{\sqrt{3}}} \frac{du}{u^2+1}\\ &= \frac{\sqrt{3}}{6}\left(\arctan\left(\frac{5}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right)\end{aligned}$$

Maintenant

$$\begin{aligned}\tan\left(\arctan\left(\frac{5}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right) &= \frac{\frac{5}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{1+\frac{5}{\sqrt{3}}\times\frac{1}{\sqrt{3}}}\\ &= \frac{\frac{4}{\sqrt{3}}}{1+\frac{5}{3}}\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

Donc $\arctan\left(\frac{5}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right) = \arctan\left(\dfrac{\sqrt{3}}{2}\right) + k\pi$. *D'après la propriété des accroisse-*

ments finis, $\arctan\left(\frac{5}{\sqrt{3}}\right)-\arctan\left(\frac{1}{\sqrt{3}}\right)\leqslant \frac{5}{\sqrt{3}}-\frac{1}{\sqrt{3}}$,

donc $\left|\arctan\left(\frac{5}{\sqrt{3}}\right)-\arctan\left(\frac{1}{\sqrt{3}}\right)-\arctan\left(\frac{\sqrt{3}}{2}\right)\right|\leqslant \frac{4}{\sqrt{3}}+\frac{\sqrt{3}}{2}<\pi$ *d'où* $k=0$ *et*

$$\int_0^2 \frac{x\,dx}{(x+1)(x^2+x+1)} = -\ln 3+\frac{1}{2}\ln 7+\frac{\sqrt{3}}{6}\arctan\left(\frac{\sqrt{3}}{2}\right).$$

7.

$$\begin{aligned}\int_0^1 \frac{dx}{(x^2+x+1)^2} &= \int_0^1 \frac{dx}{\left((x+\frac{1}{2})^2+\frac{3}{4}\right)^2}\\ &= \frac{16}{9}\int_0^1 \frac{dx}{\left((\frac{2x+1}{\sqrt{3}})^2+1\right)^2}\\ &= \frac{16}{9}\times\frac{\sqrt{3}}{2}\int_{\frac{1}{\sqrt{3}}}^{\frac{3}{\sqrt{3}}} \frac{du}{(u^2+1)^2}\end{aligned}$$

En posant $u=\tan\varphi$, $d\varphi=\dfrac{du}{u^2+1}$,

$$\begin{aligned}\int_0^1 \frac{dx}{(x^2+x+1)^2} &= \frac{8\sqrt{3}}{9}\int_{\arctan\left(\frac{1}{\sqrt{3}}\right)}^{\arctan\left(\frac{3}{\sqrt{3}}\right)} \cos^2\varphi\,d\varphi\\ &= \frac{4\sqrt{3}}{9}\left[\varphi+\frac{1}{2}\sin 2\varphi\right]_{\arctan\left(\frac{1}{\sqrt{3}}\right)}^{\arctan\left(\frac{3}{\sqrt{3}}\right)}\\ &= \frac{4\sqrt{3}}{9}\left(\arctan\left(\frac{3}{\sqrt{3}}\right)-\arctan\left(\frac{1}{\sqrt{3}}\right)\right)\\ &\quad+\frac{2\sqrt{3}}{9}\left(\sin\left(2\arctan\left(\frac{3}{\sqrt{3}}\right)\right)-\sin\left(2\arctan\left(\frac{1}{\sqrt{3}}\right)\right)\right).\end{aligned}$$

8.

$$\frac{1}{x-2}-\frac{1}{x-1}=\frac{1}{(x-2)(x-1)}.$$

En élevant au carré,

$$\frac{1}{(x^2-3x+2)^2}=\frac{1}{(x-2)^2}+\frac{1}{(x-1)^2}-\frac{2}{(x-2)(x-1)}=\frac{1}{(x-2)^2}+\frac{1}{(x-1)^2}-\frac{2}{x-2}+\frac{2}{x-1}.$$

D'où

$$\begin{aligned}\int_0^{-1} \frac{dx}{(x^2-3x+2)^2} &= \left[-\frac{1}{x-2}-\frac{1}{x-1}-\ln(2-x)+\ln(1-x)\right]_0^{-1}\\ &= \frac{1}{3}+\frac{1}{2}+\ln 2-\frac{1}{2}-1+\ln 2\\ &= -\frac{2}{3}+2\ln 2.\end{aligned}$$

9. Le lecteur vérifiera que

$$\int \frac{(x-1)\,dx}{(x^2+1)^2(x+2)} = -\frac{3}{25}\log\left(|x+2|\right) + \frac{3}{50}\ln\left(x^2+1\right) - \frac{17}{50}\mathrm{arctg}\,(x) - \frac{1}{10}\frac{x+3}{x^2+1} + C$$

et que

$$\int_0^1 \frac{(x-1)\,dx}{(x^2+1)^2(x+2)} = \frac{3}{25}\ln 3 + \frac{9}{50}\ln 2 - \frac{17\pi}{200} + \frac{1}{10}.$$

Références